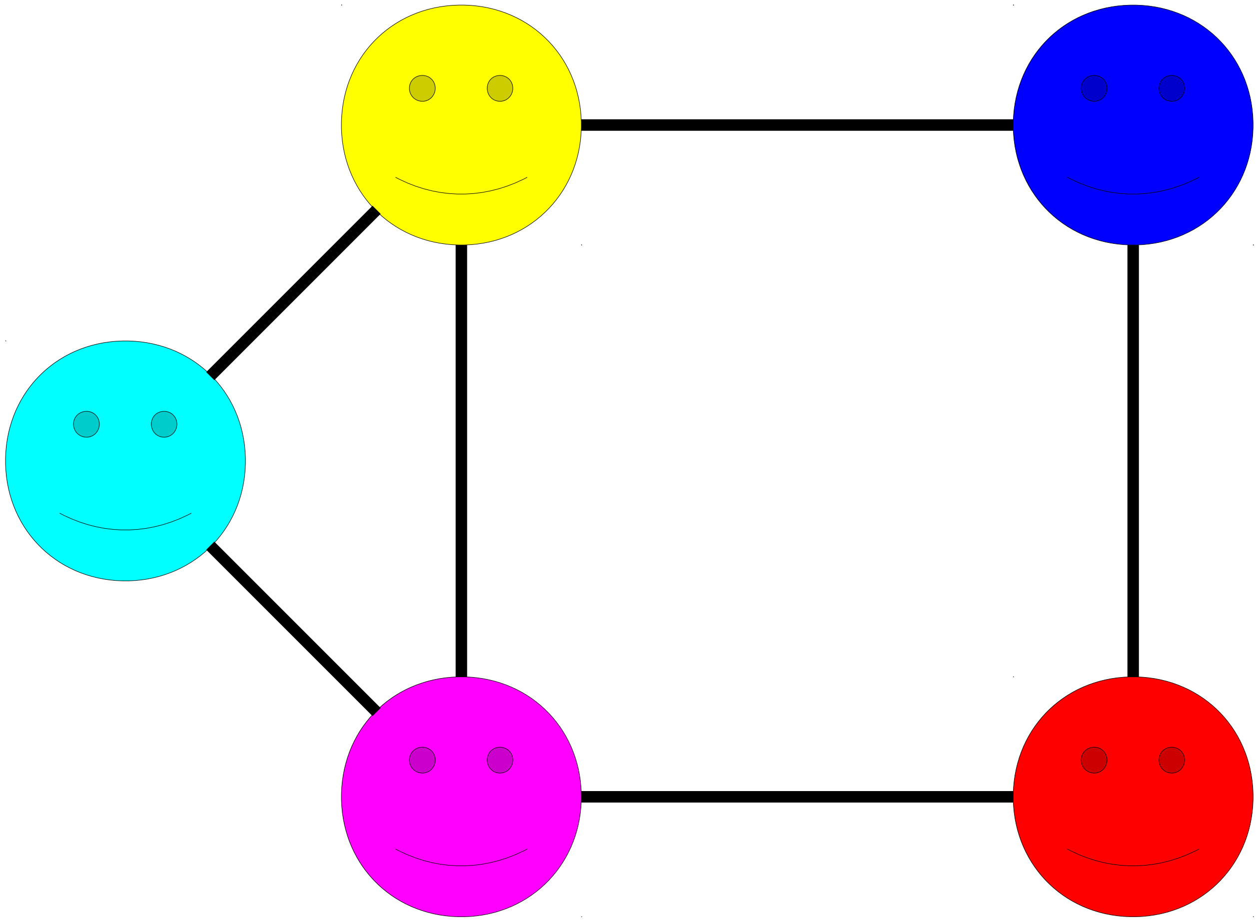
Graph Theory

Part Two

Recap from Last Time

A ***graph*** is a mathematical structure for representing relationships.



A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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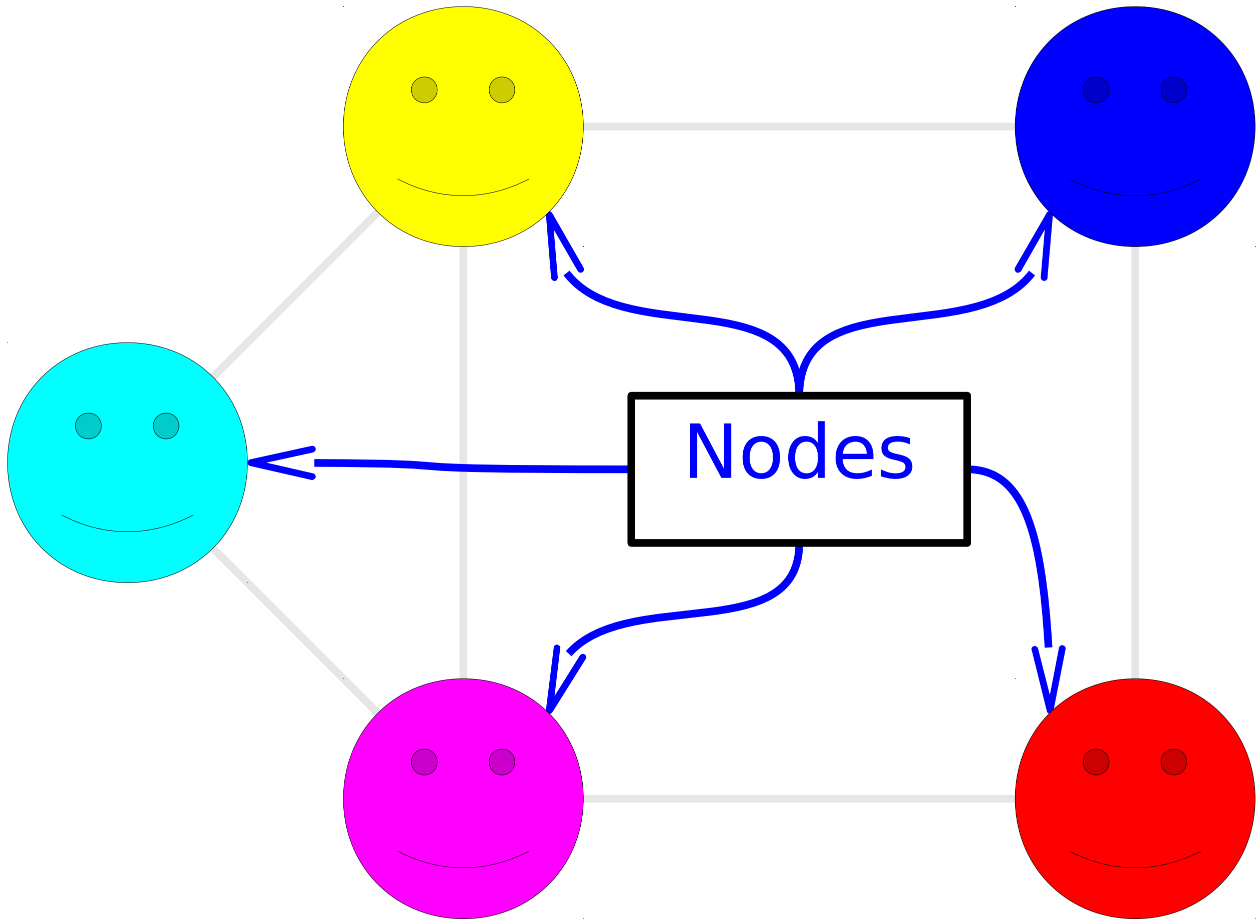

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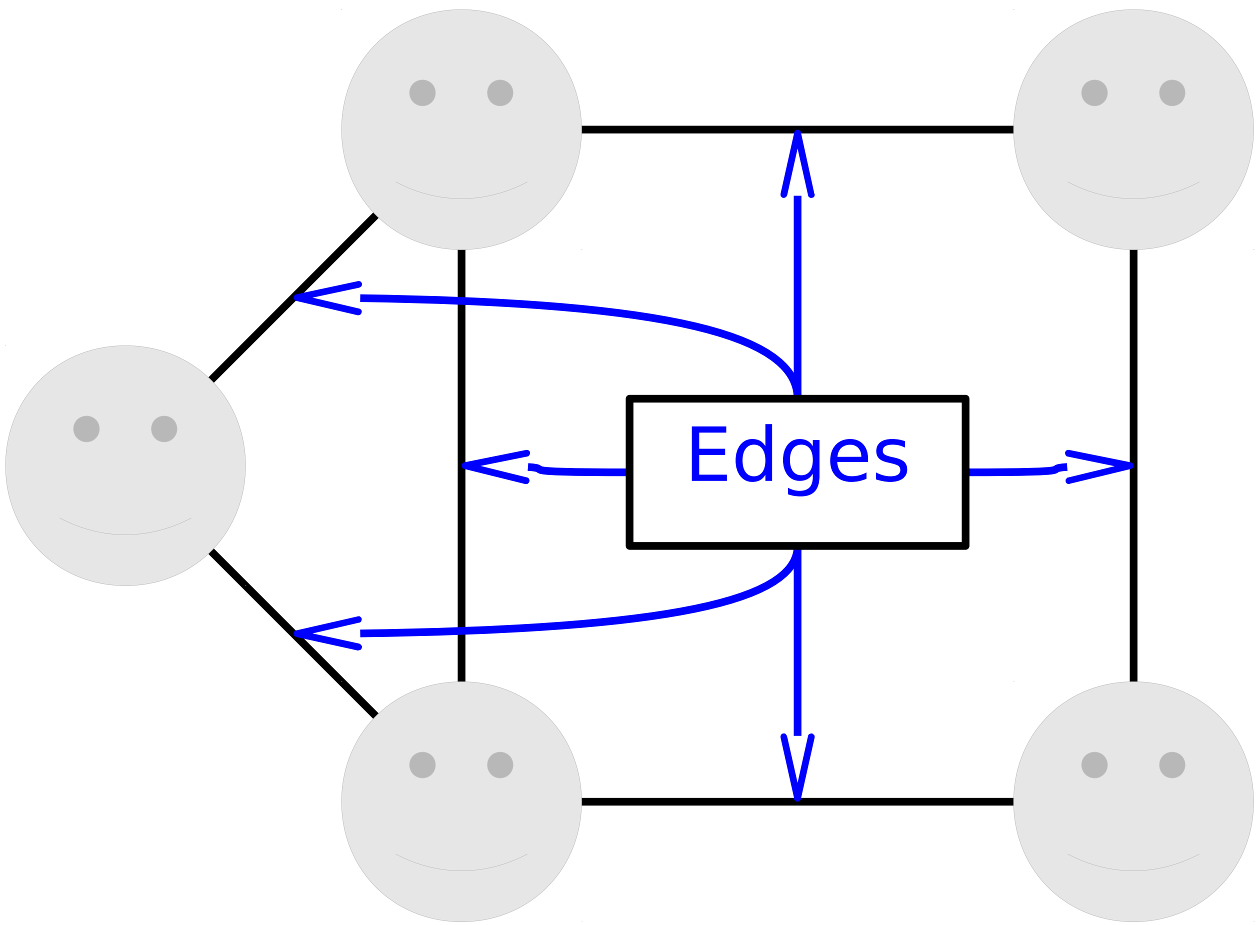

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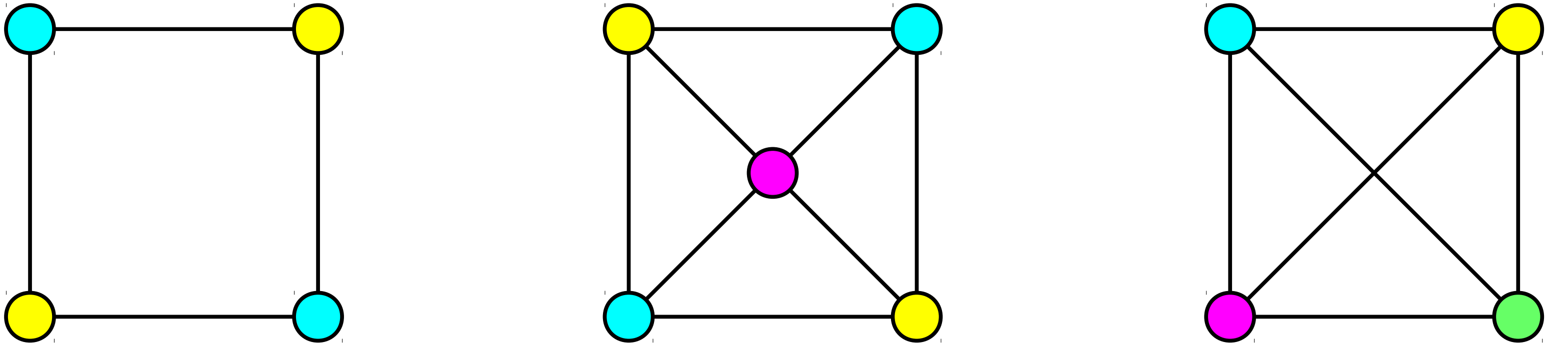
Adjacency and Connectivity

- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***connected*** if there's a path between them.
  - A path is a series of one or more nodes where consecutive nodes are adjacent.

k-Vertex-Colorings

- If $G = (V, E)$ is a graph, a ***k-vertex-coloring*** of G is a way of assigning colors to the nodes of G, using at most k colors, so that no two nodes of the same color are adjacent.



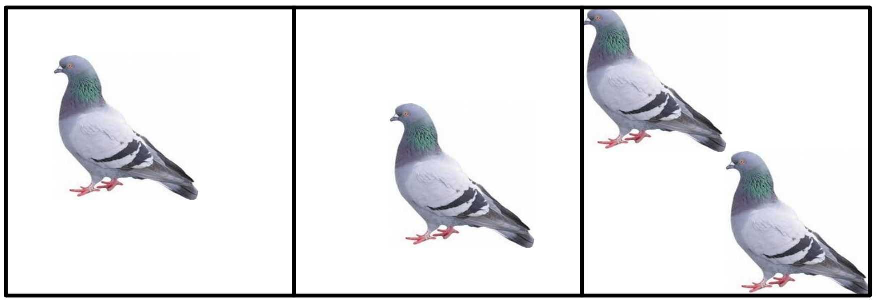
- The ***chromatic number*** of G, denoted $\boldsymbol{\chi(G)}$, is the minimum number of colors needed in any k-coloring of G.
- Today, we're going to see several results involving coloring parts of graphs. They don't necessarily involve *k-vertex-colorings* of graphs, so feel free to ask for clarifications if you need them!

New Stuff!

The Pigeonhole Principle

The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):*** If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



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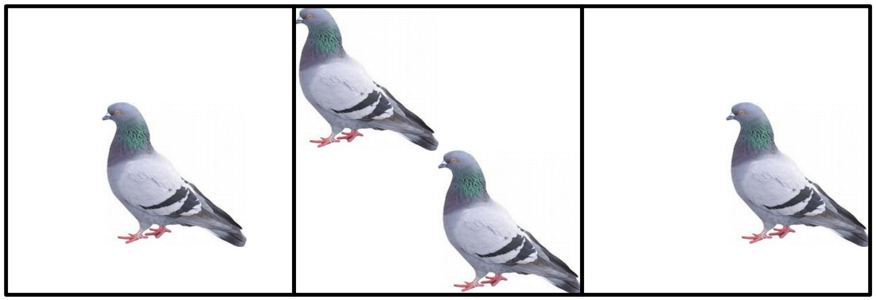

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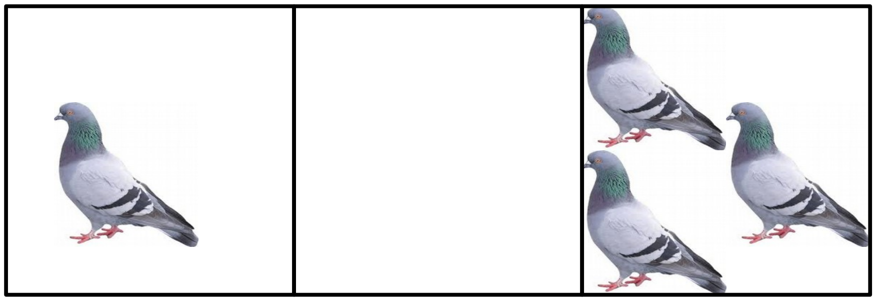

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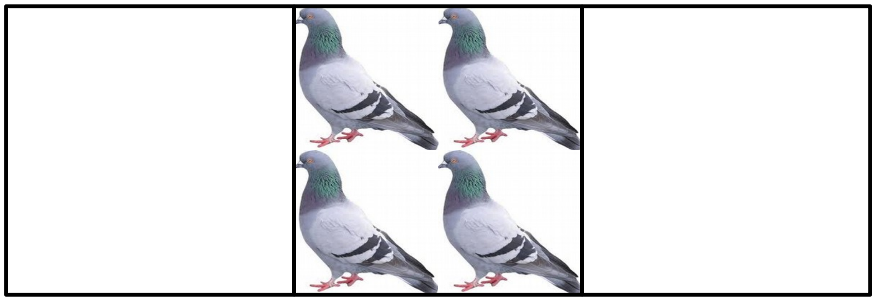


$m = 4, n = 3$

Thanks to Amy Liu for this awesome drawing!

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. **How many** of the following statements are true?

If $f : A \to B$, then f is injective.
If $f : A \to B$, then f is not injective.
If $f : A \to B$, then f is surjective.
If $f : A \to B$, then f is not surjective.

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. There are m objects in total, so we know that

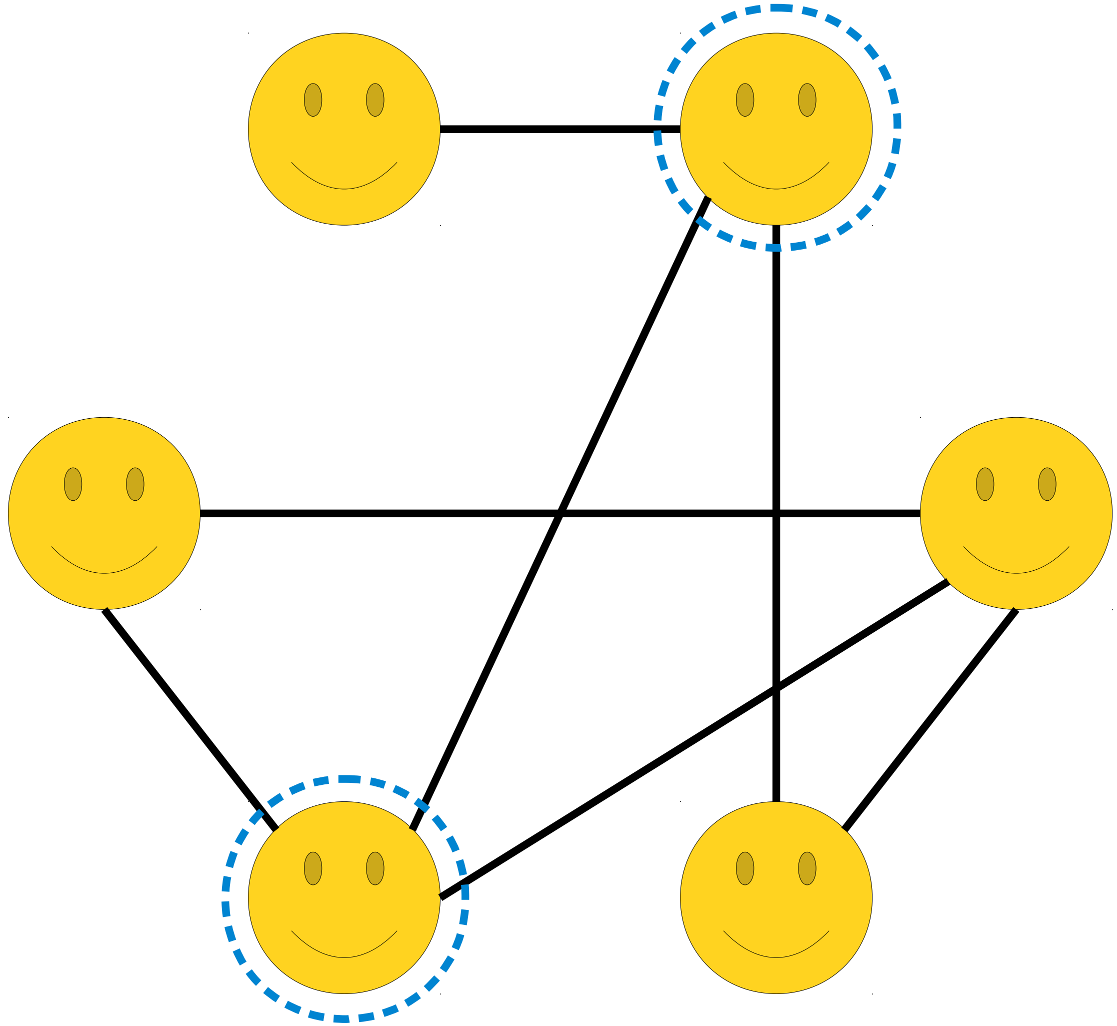
$$m = x_1 + x_2 + \ldots + x_n.$$

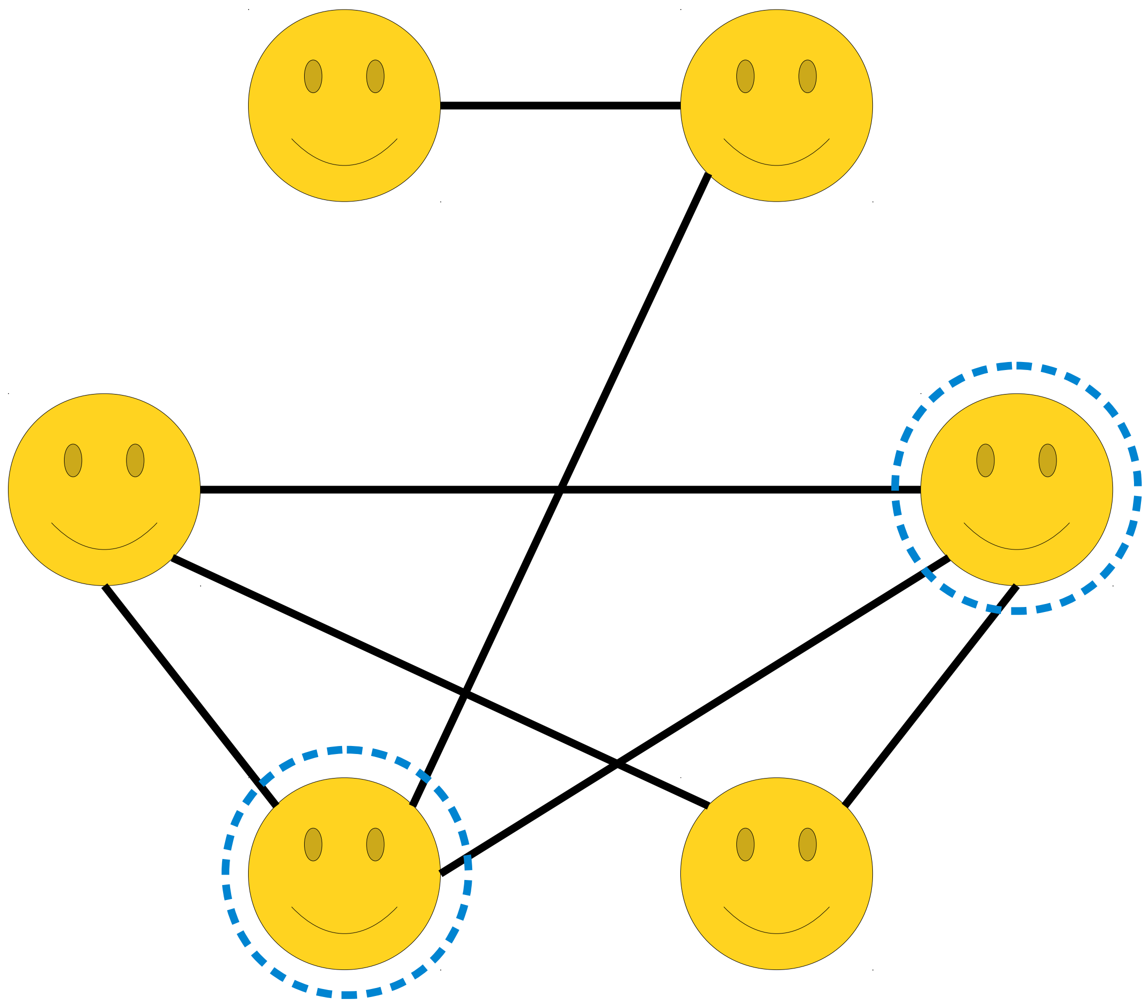
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i. This means that

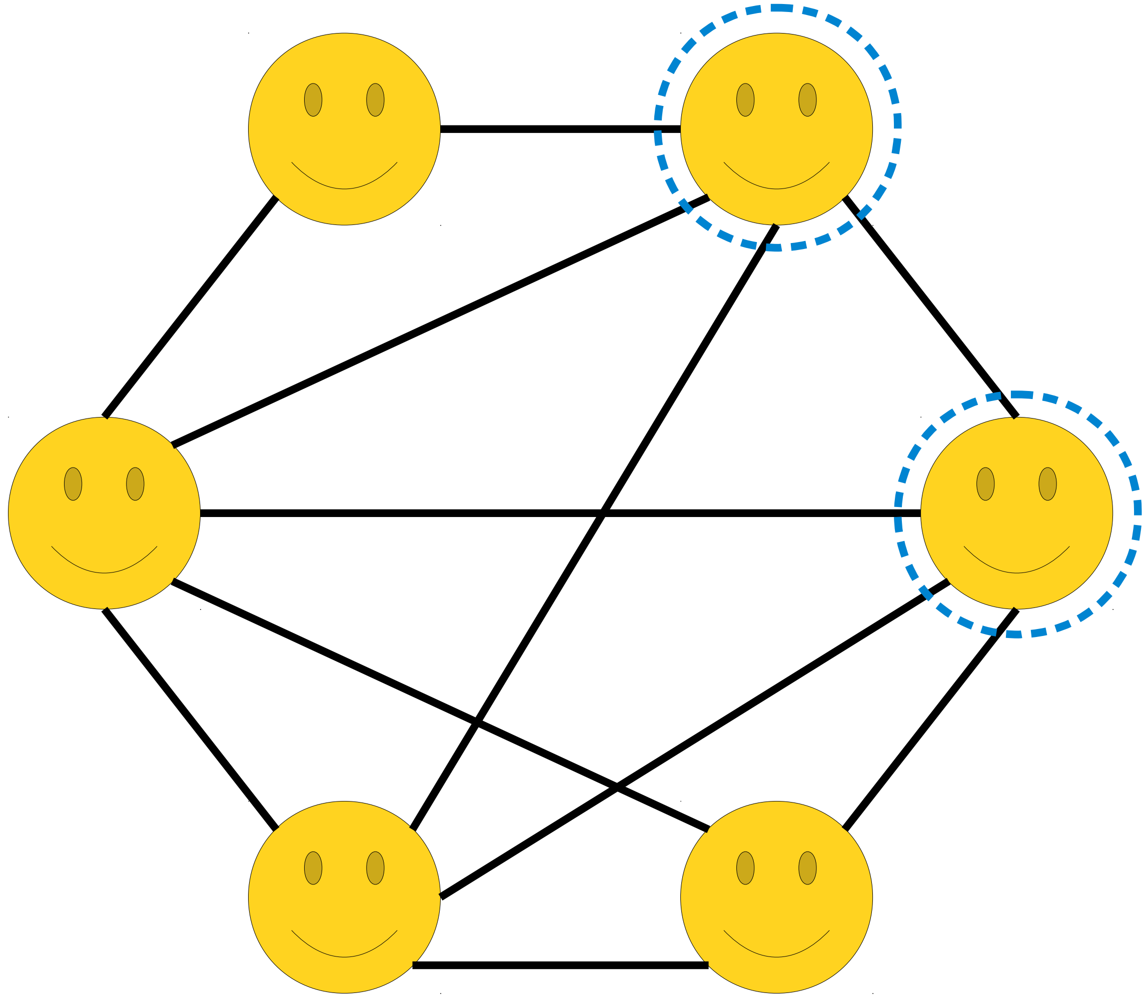
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \quad \textit{(n times)} \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks



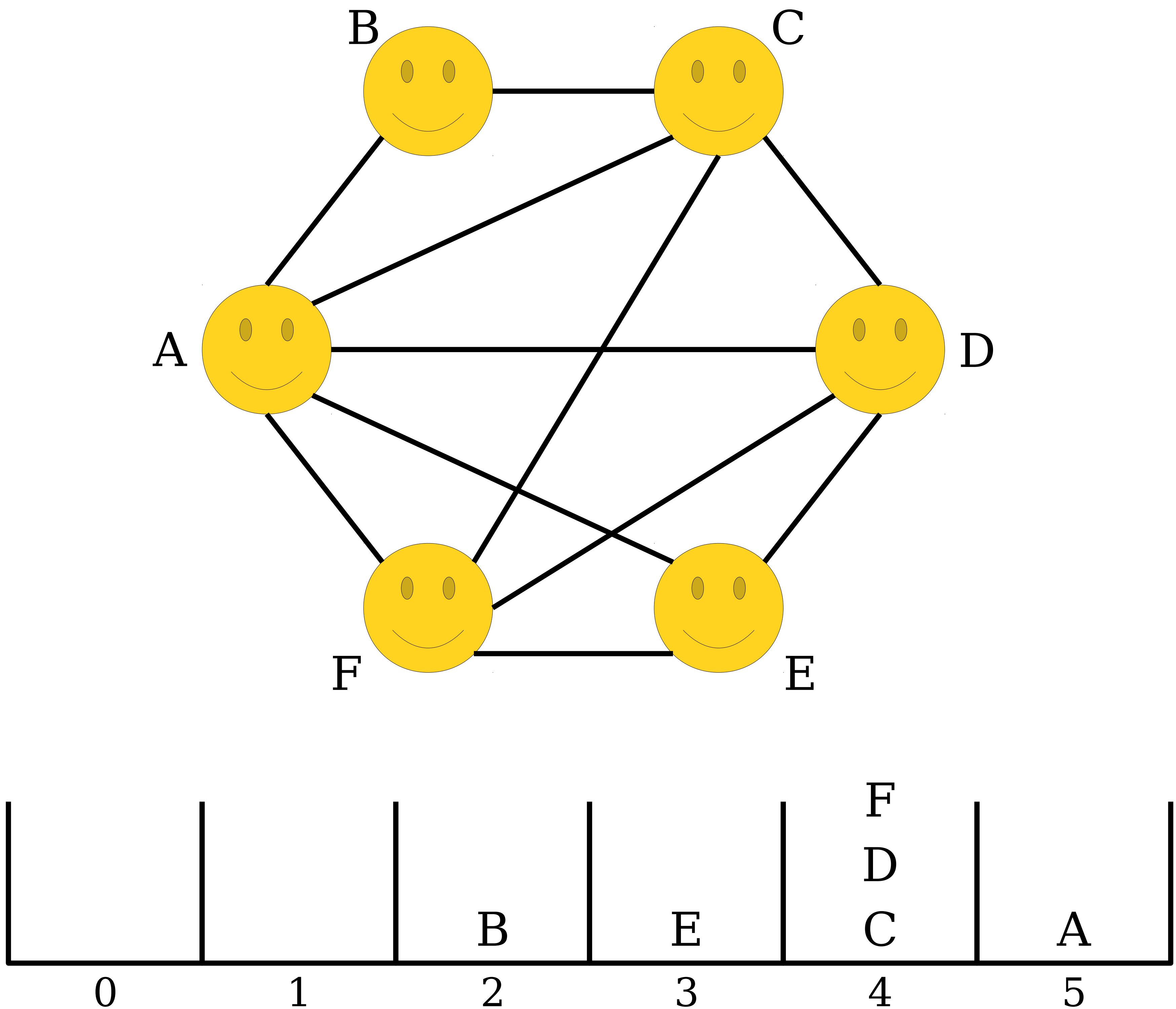




Degrees

- The ***degree*** of a node *v* in a graph is the number of nodes that *v* is adjacent to.

- ***Theorem:*** Every graph with at least two nodes has at least two nodes with the same degree.
    - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.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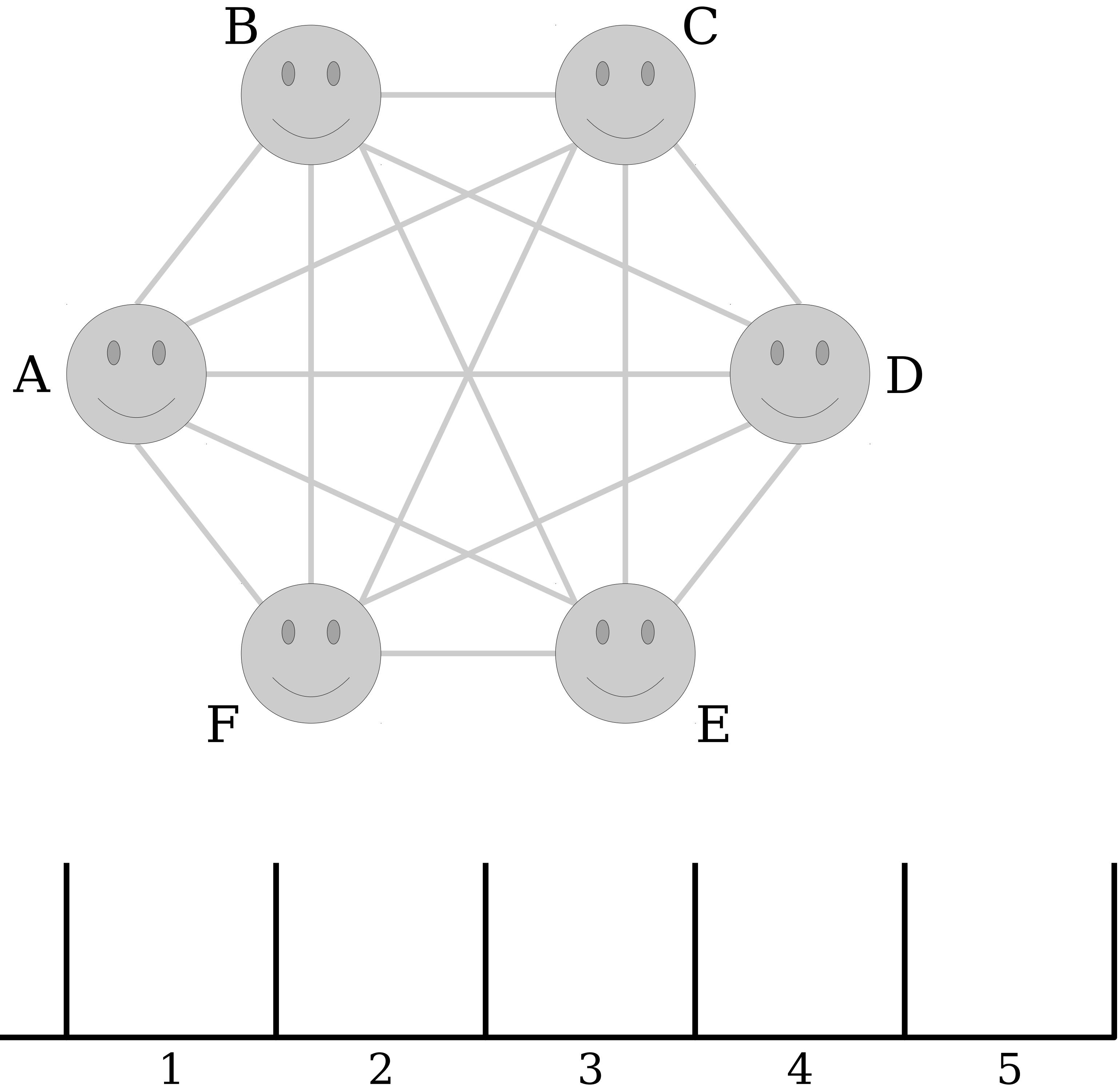
B
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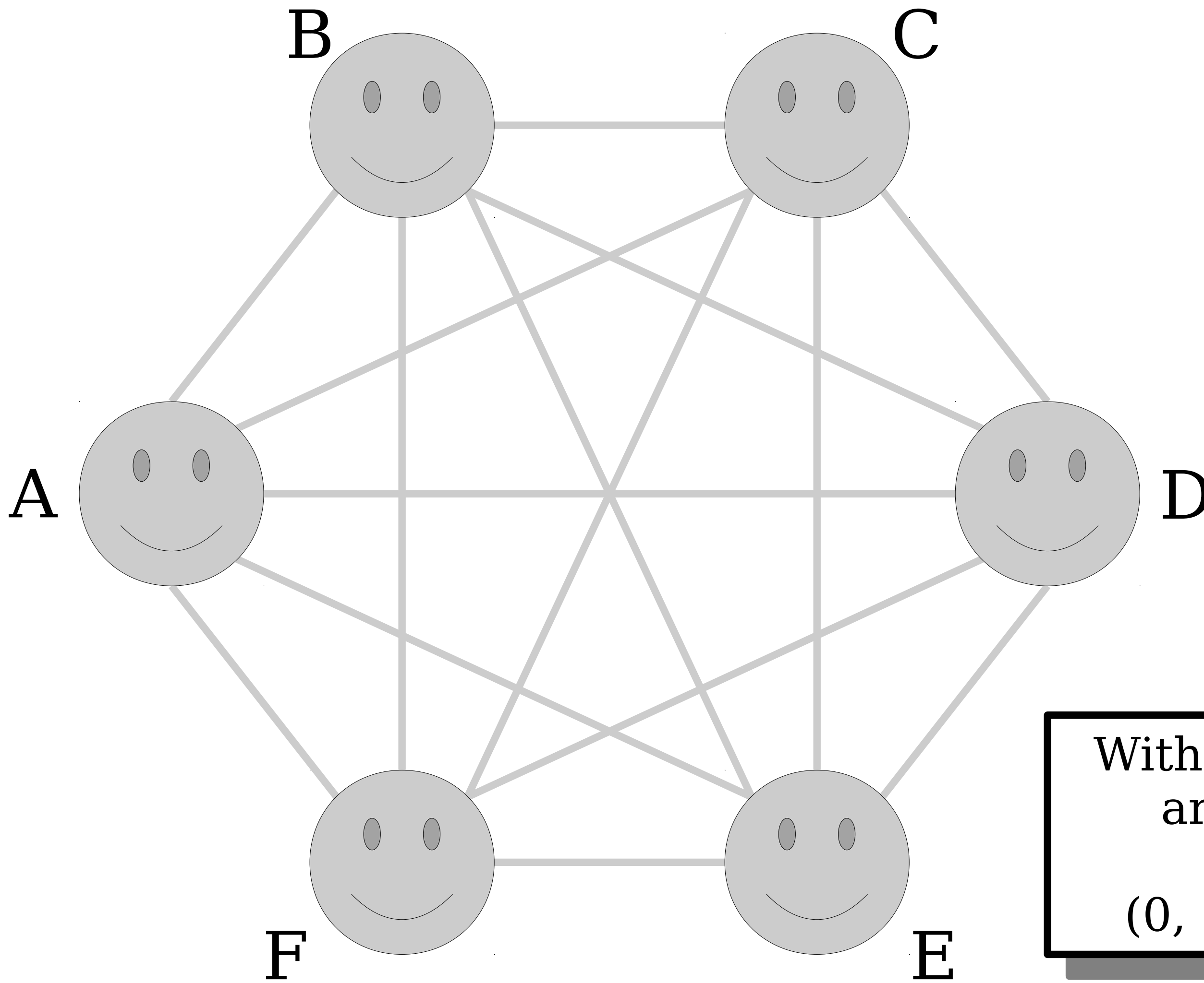
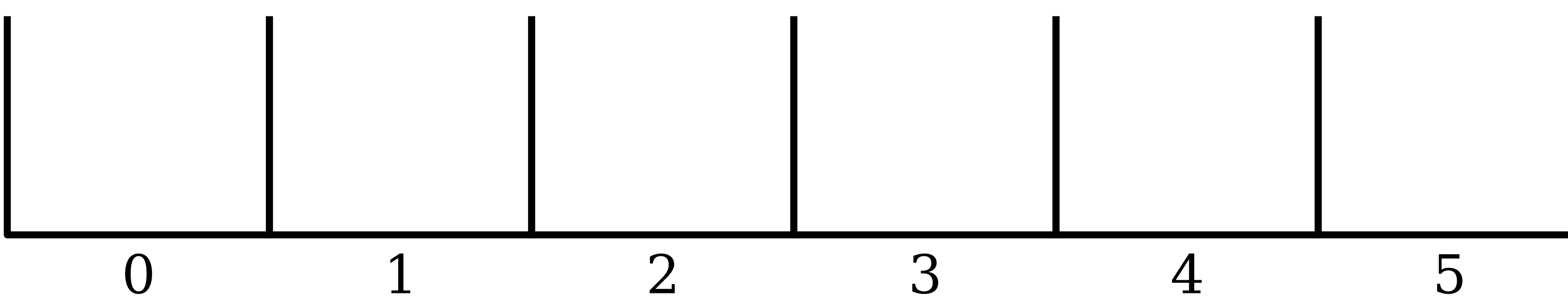
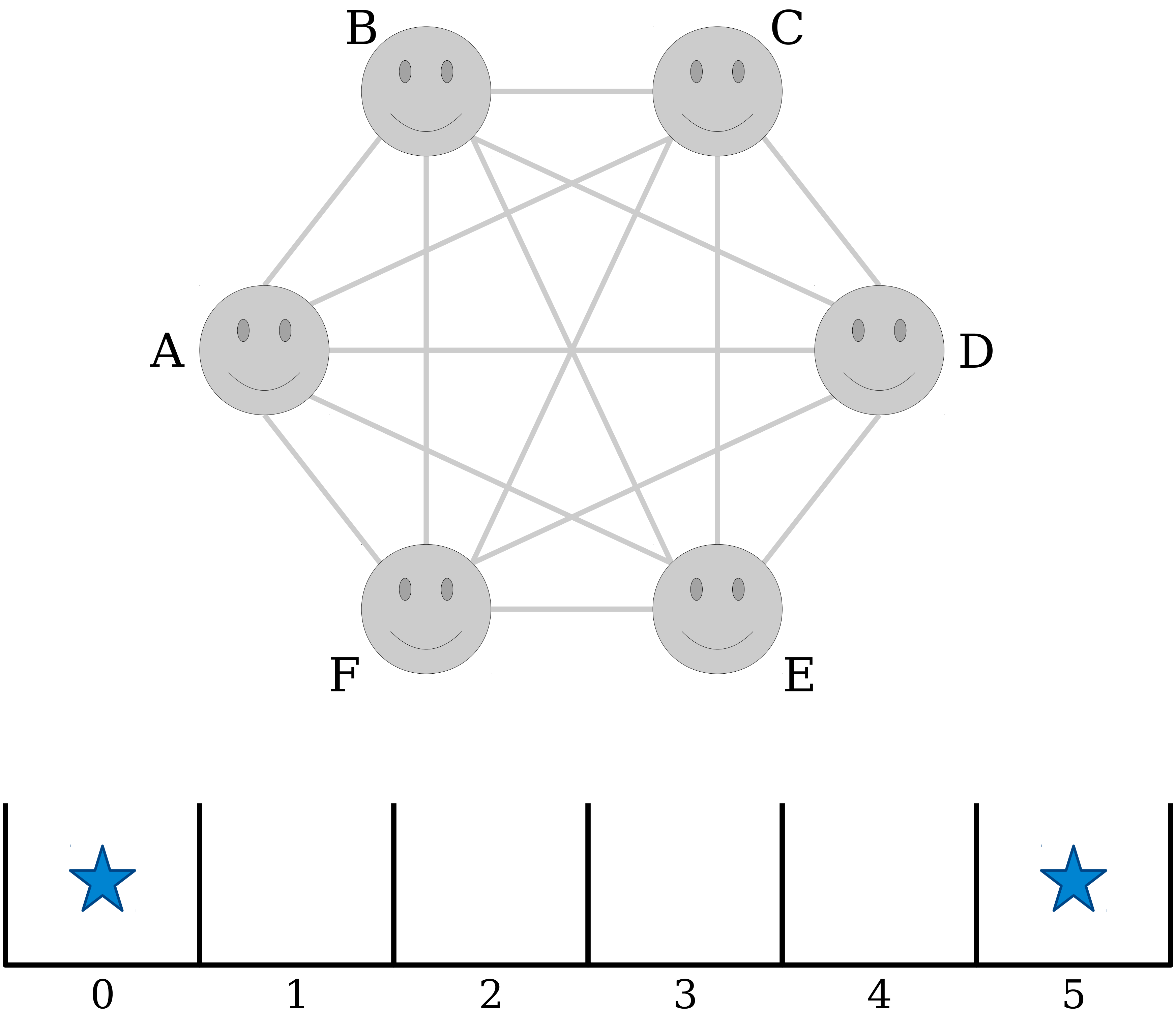


With n nodes, there are n possible degrees ($0, 1, 2, \ldots, n - 1$)

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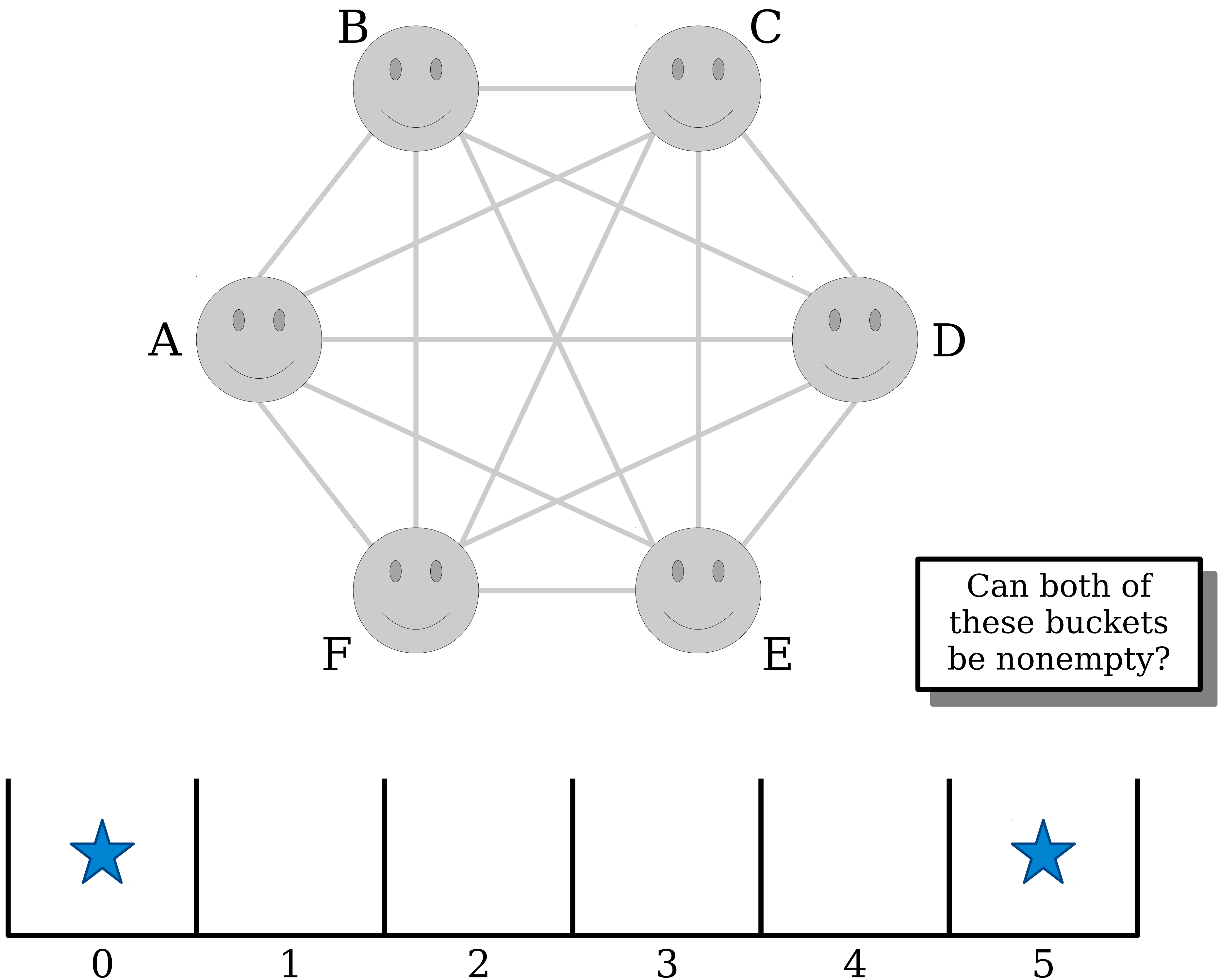
B
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Can both of these buckets be nonempty?
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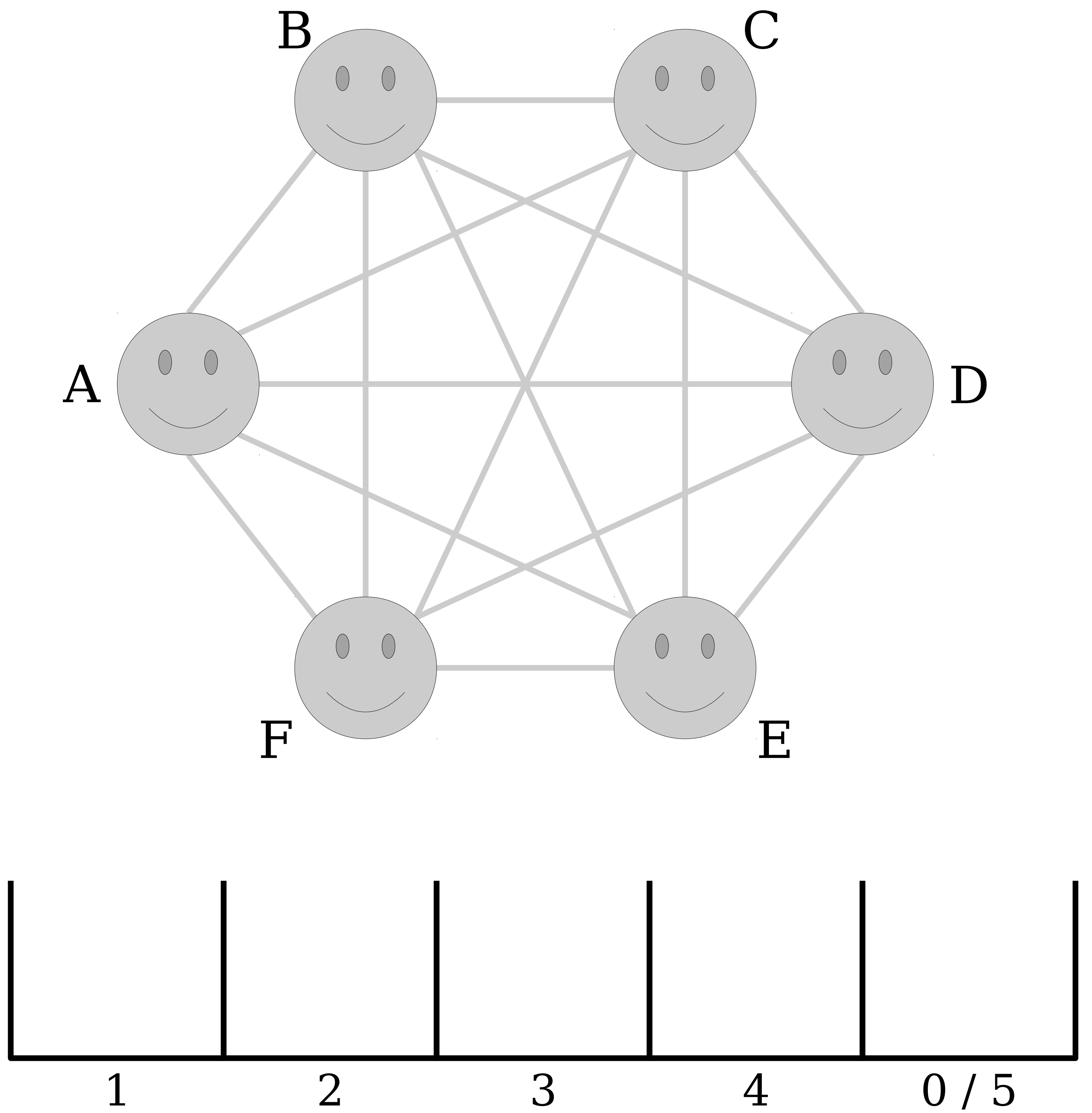
5


B
C
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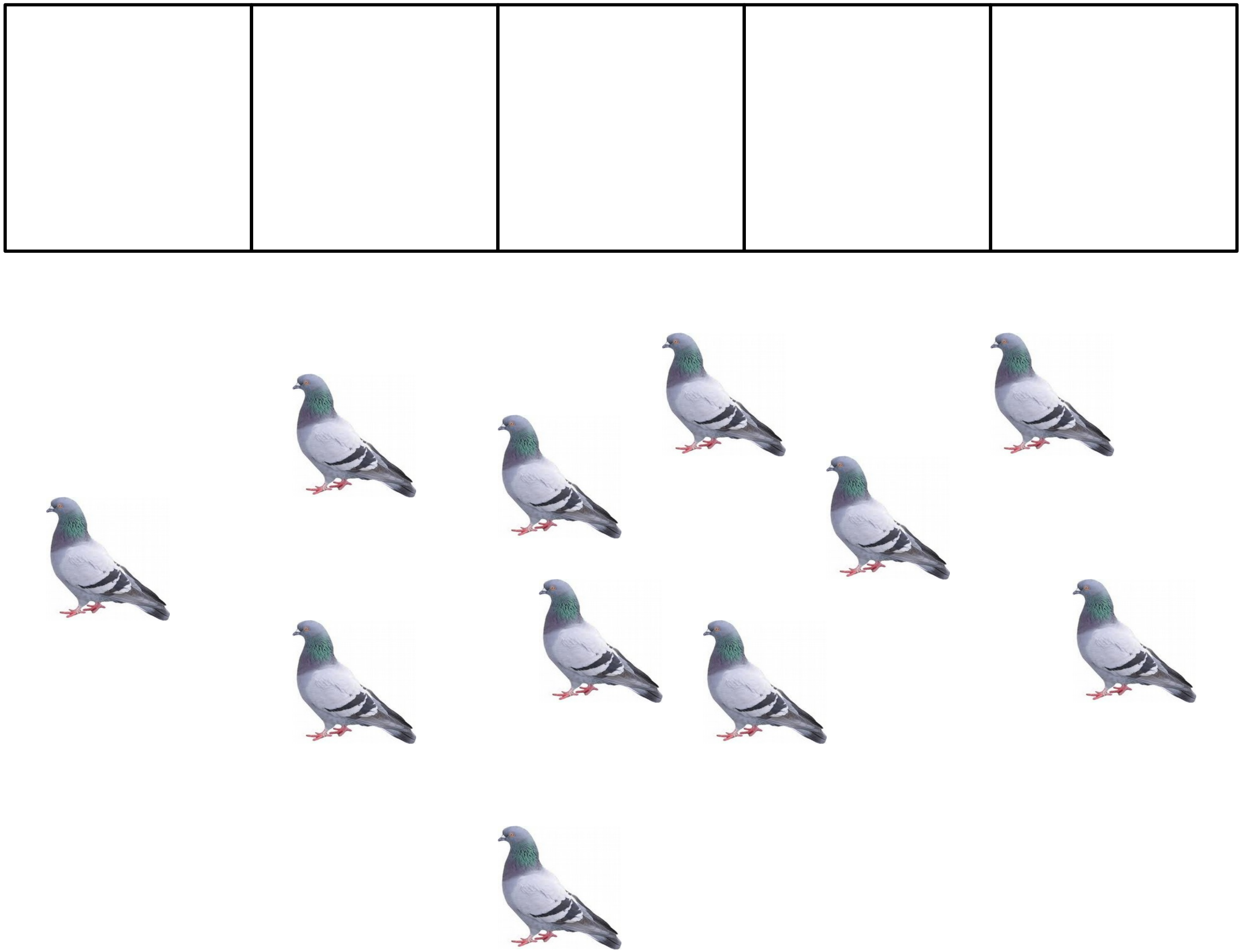
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G, namely 0, 1, 2, …, $n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

The Pigeonhole Principle



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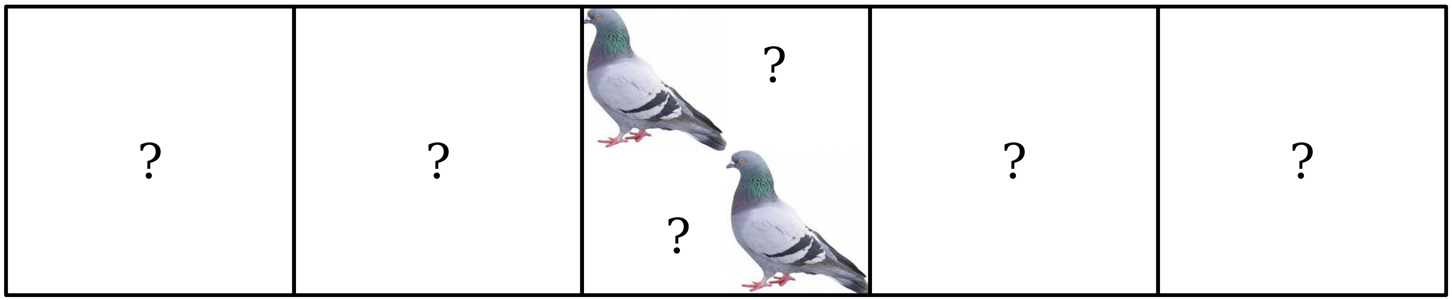

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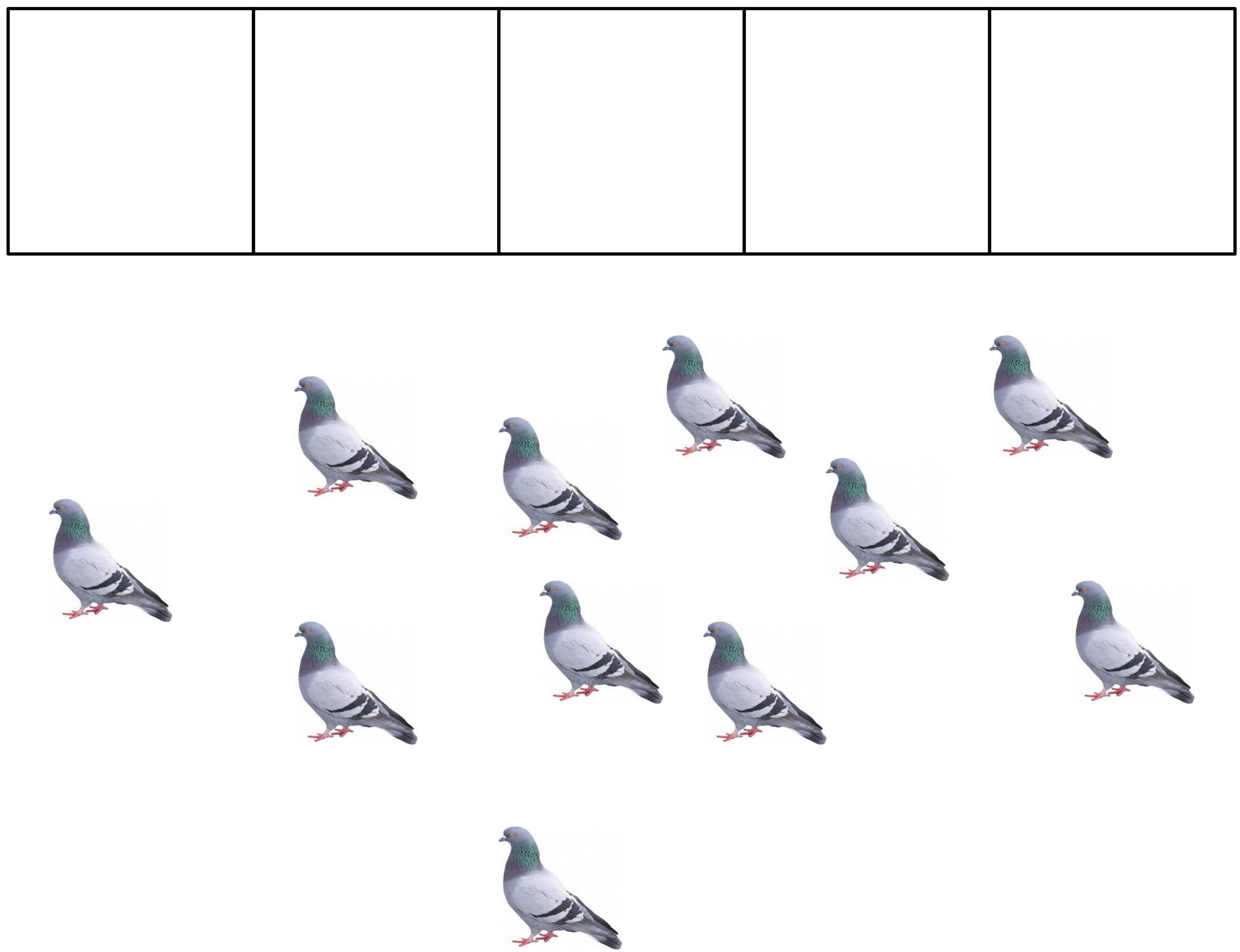

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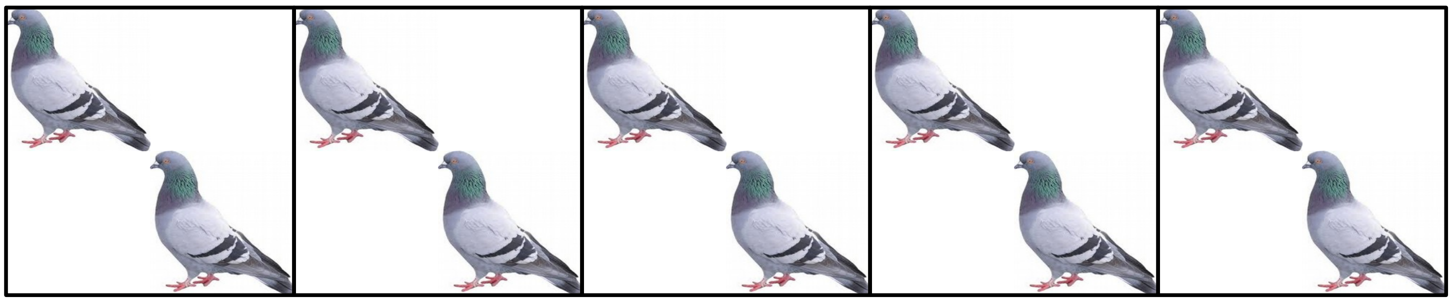

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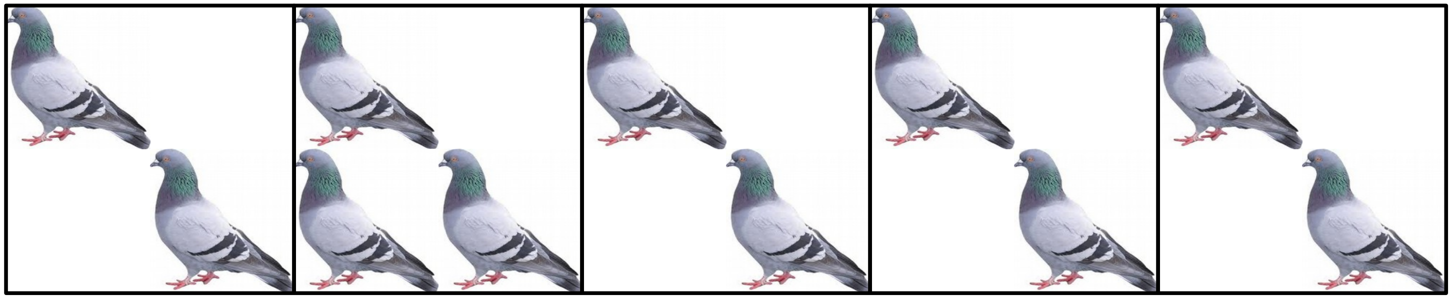

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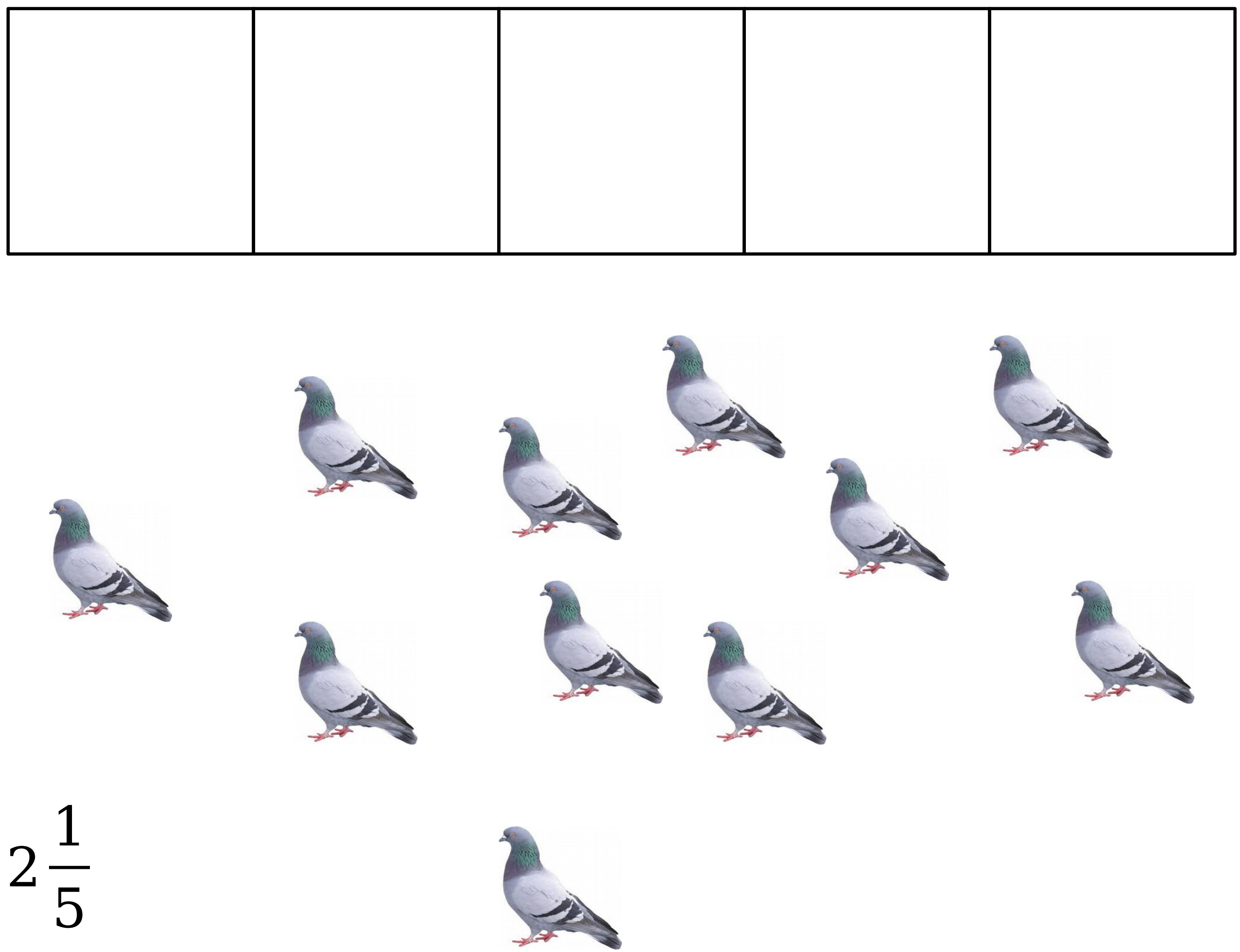

$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The ***generalized pigeonhole principle*** says that if you distribute *m* objects into *n* bins, then
  - some bin will have at least $\lceil m/n \rceil$ objects in it, and
  - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means "m/n, rounded up."
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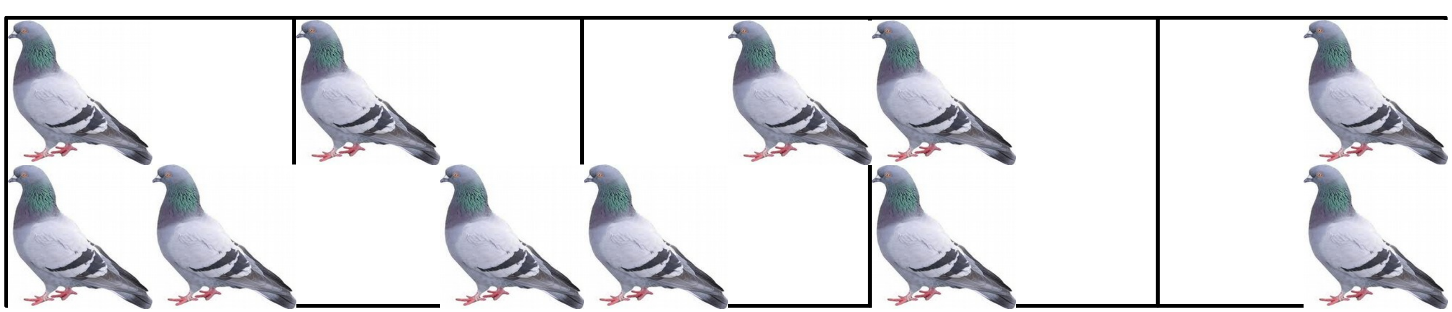

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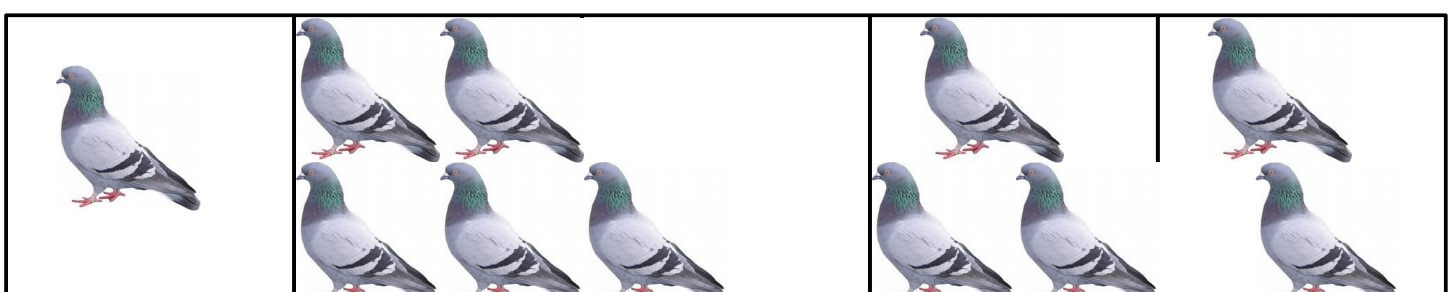

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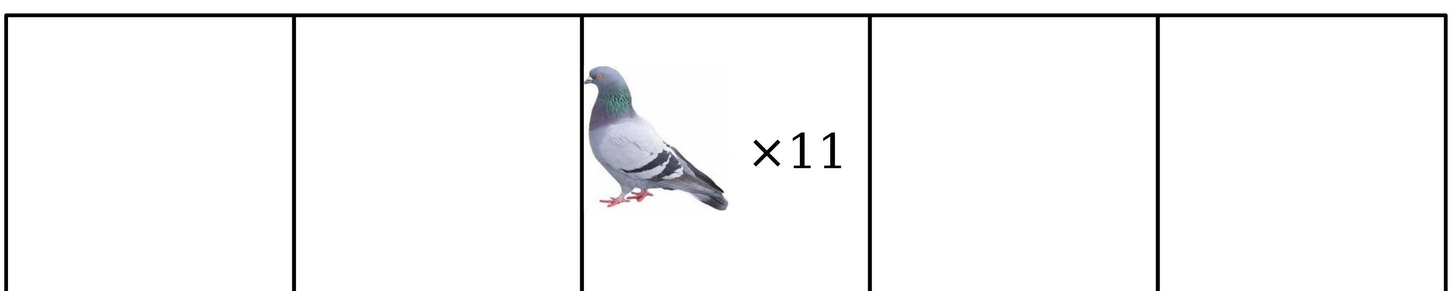

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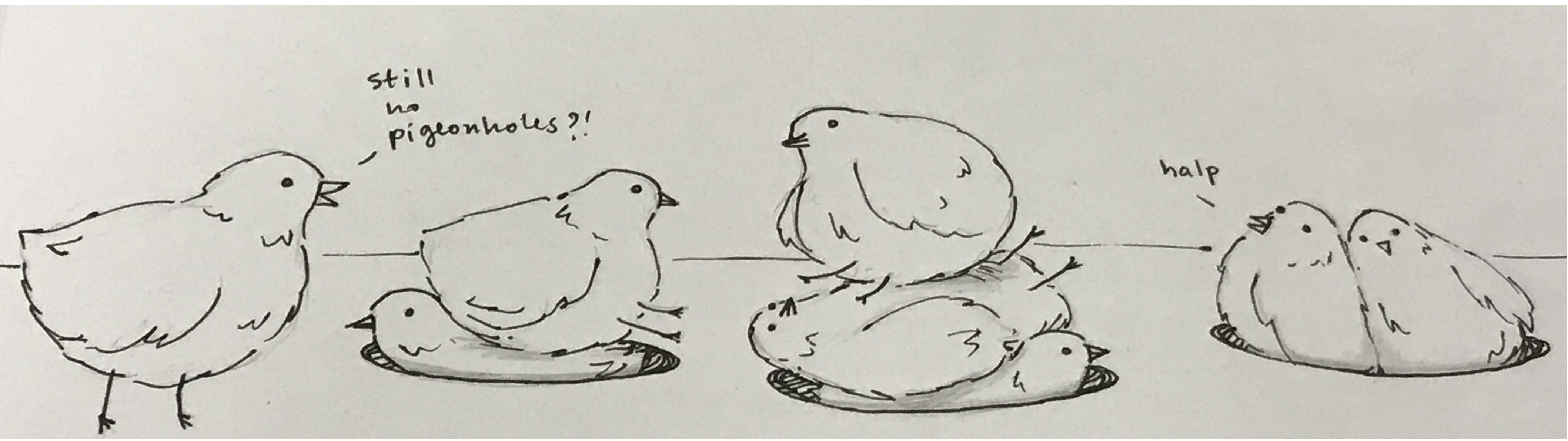


$m = 8, n = 3$

Thanks to Amy Liu for this awesome drawing!

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least m/n objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n, there is a way to distribute m objects into n bins such that each bin contains fewer than m/n objects.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. Since there are m objects in total, we know that

$$m = x_1 + x_2 + \ldots + x_n.$$

Since each bin contains fewer than m/n objects, we see that $x_i < m/n$ for each i. Therefore, we have that

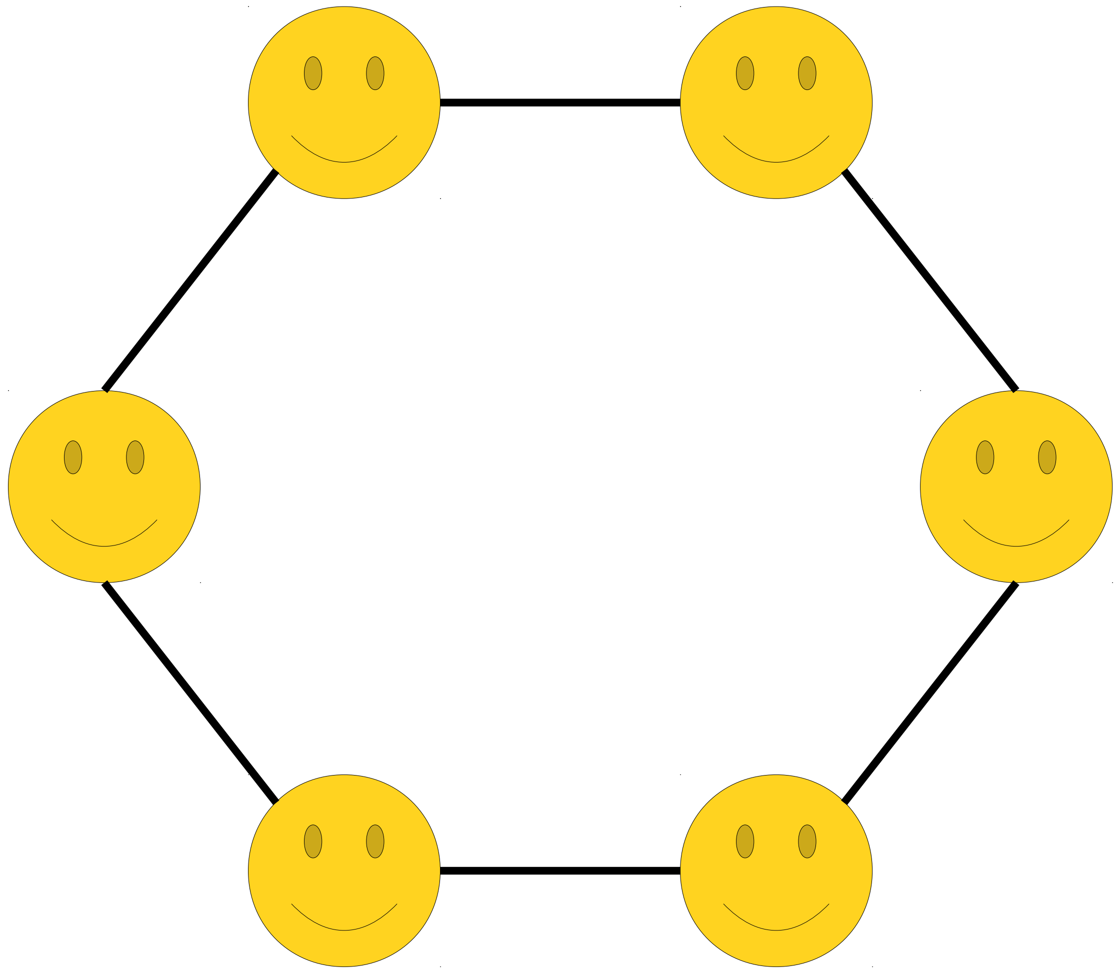
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &< m/n + m/n + \ldots + m/n \quad (n \text{ times}) \\ &= m. \end{aligned}$$

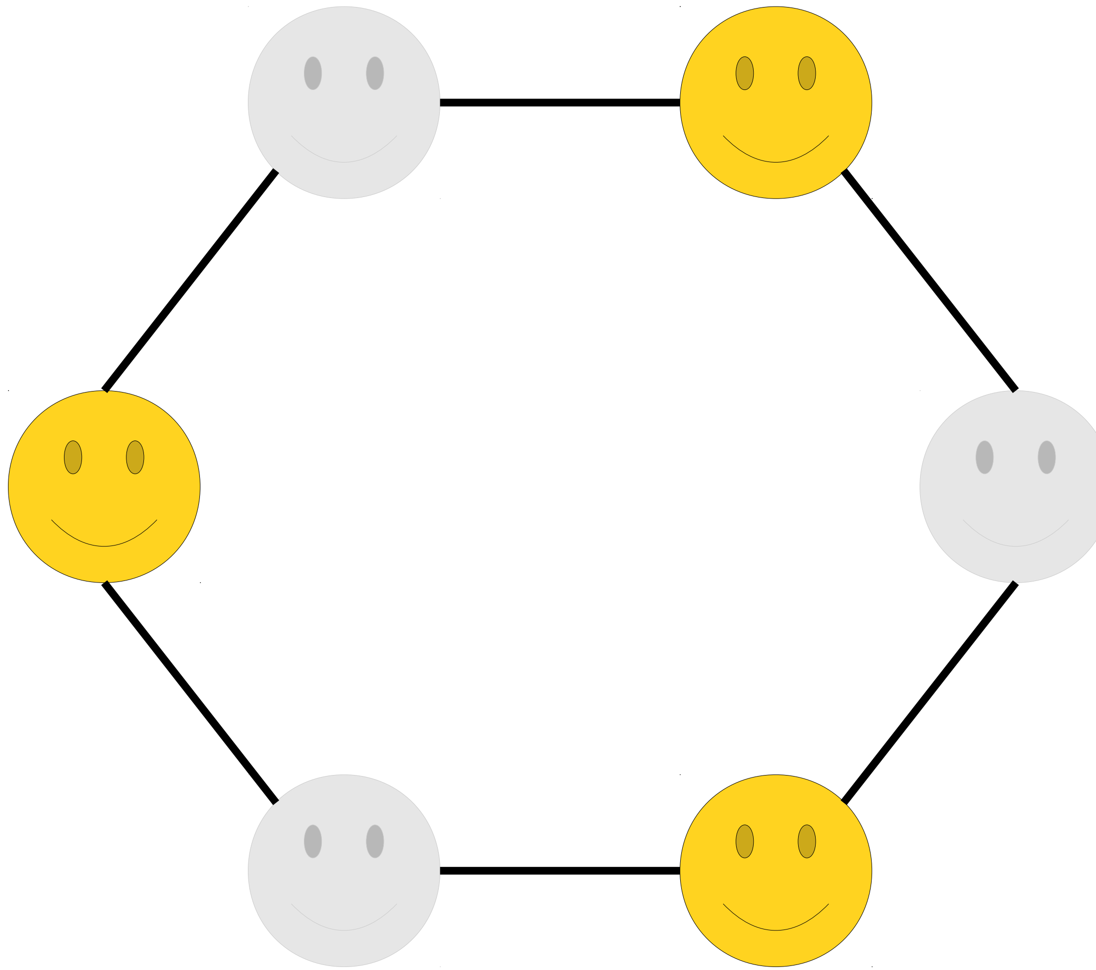
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

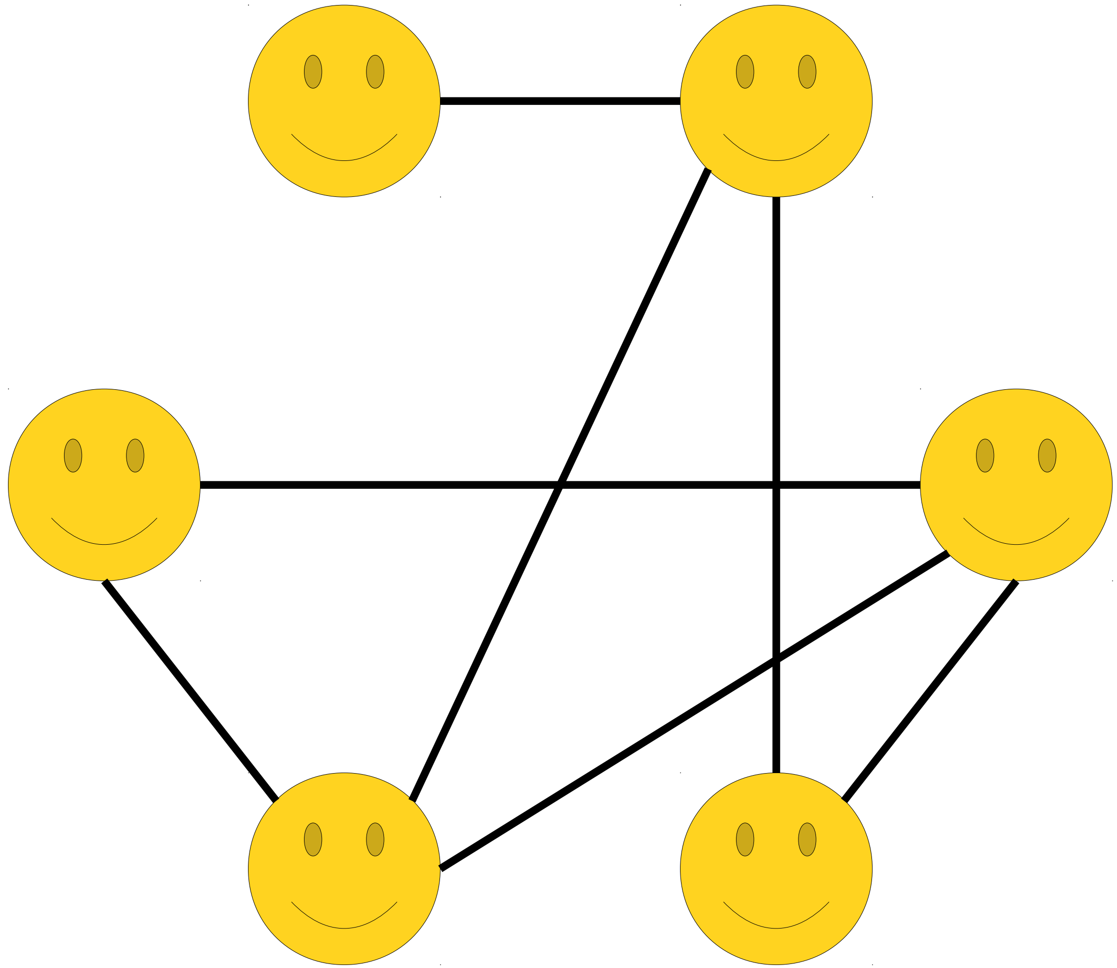
An Application: Friends and Strangers

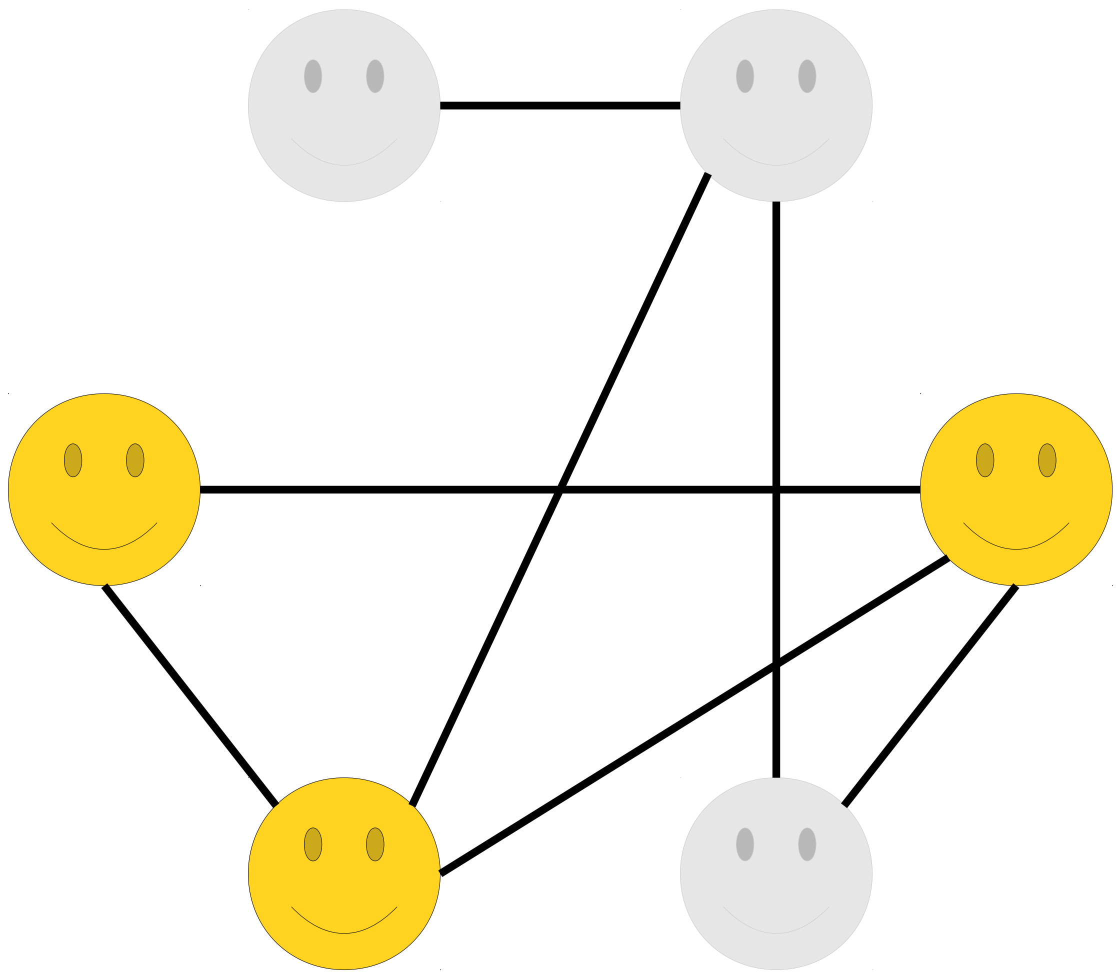
Friends and Strangers

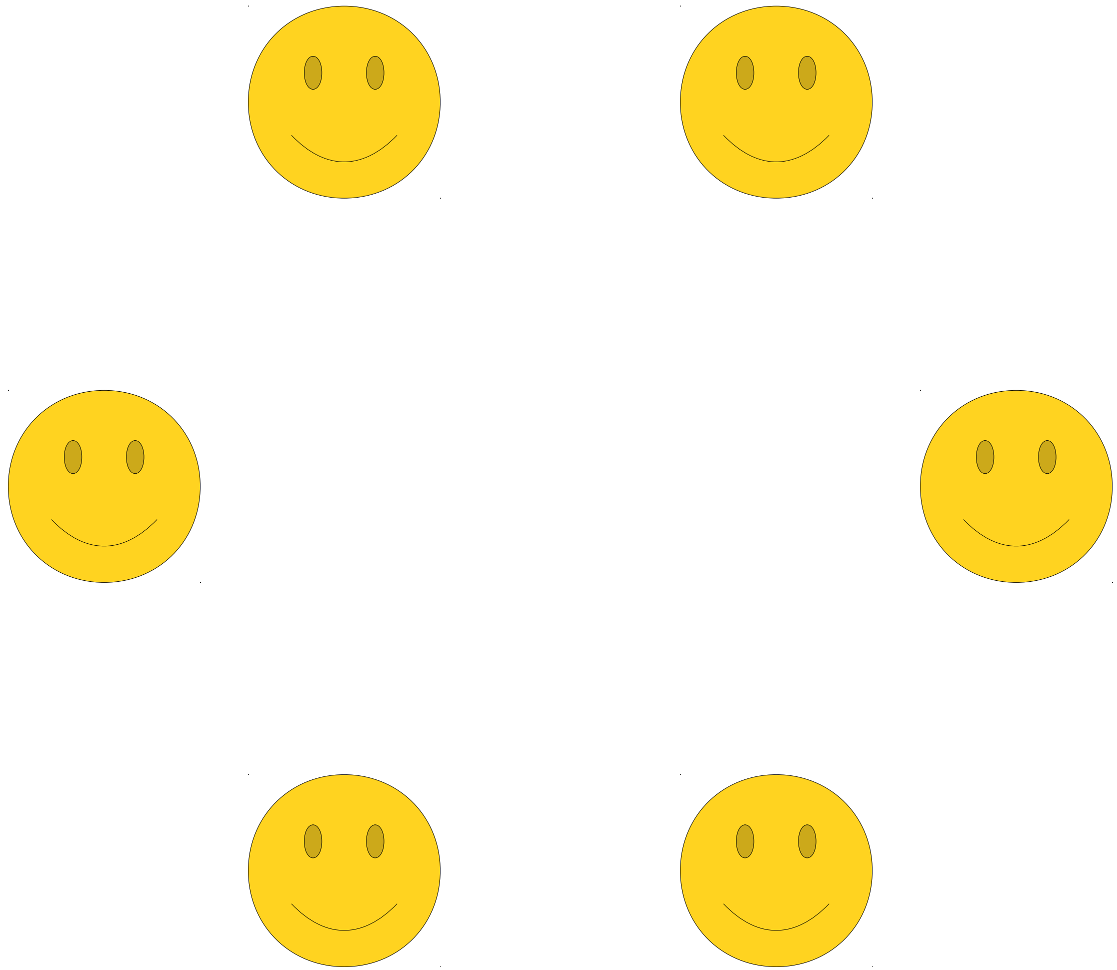
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

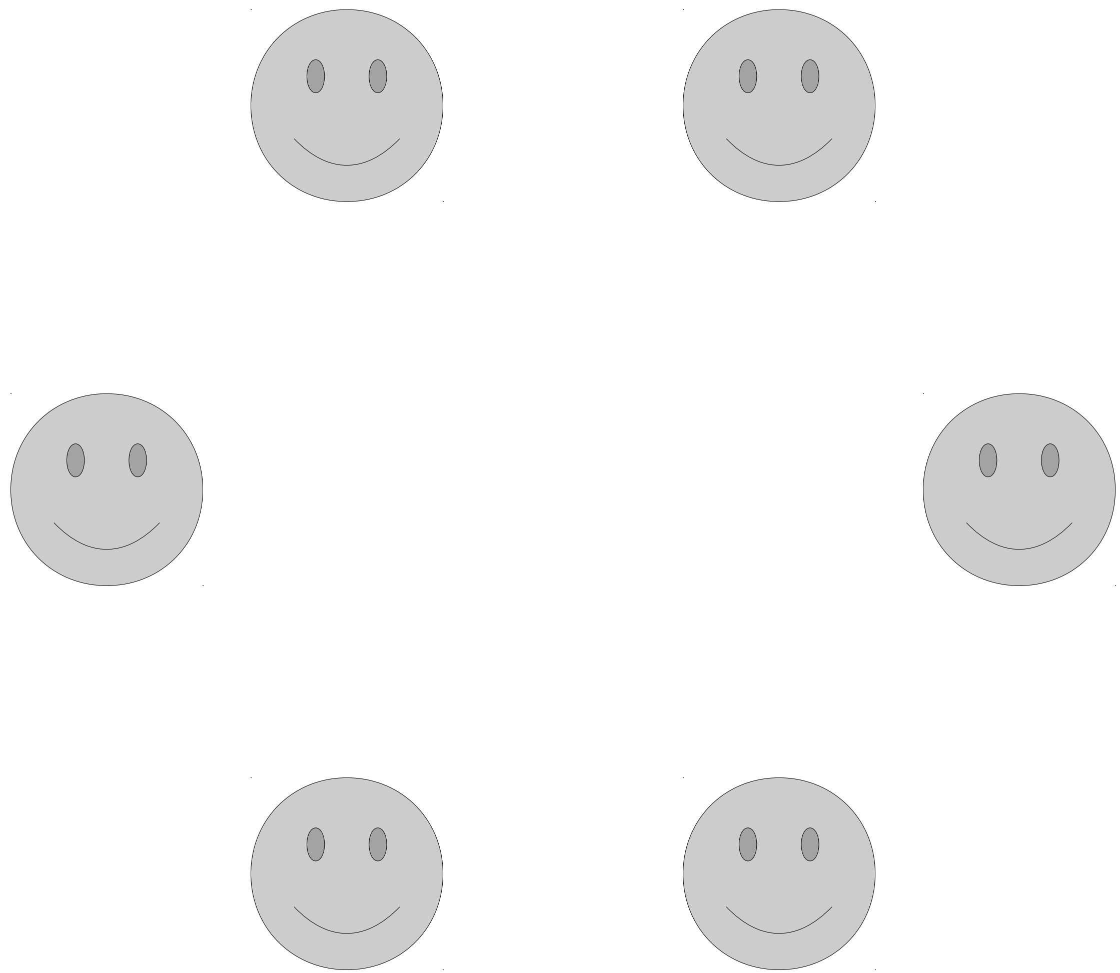


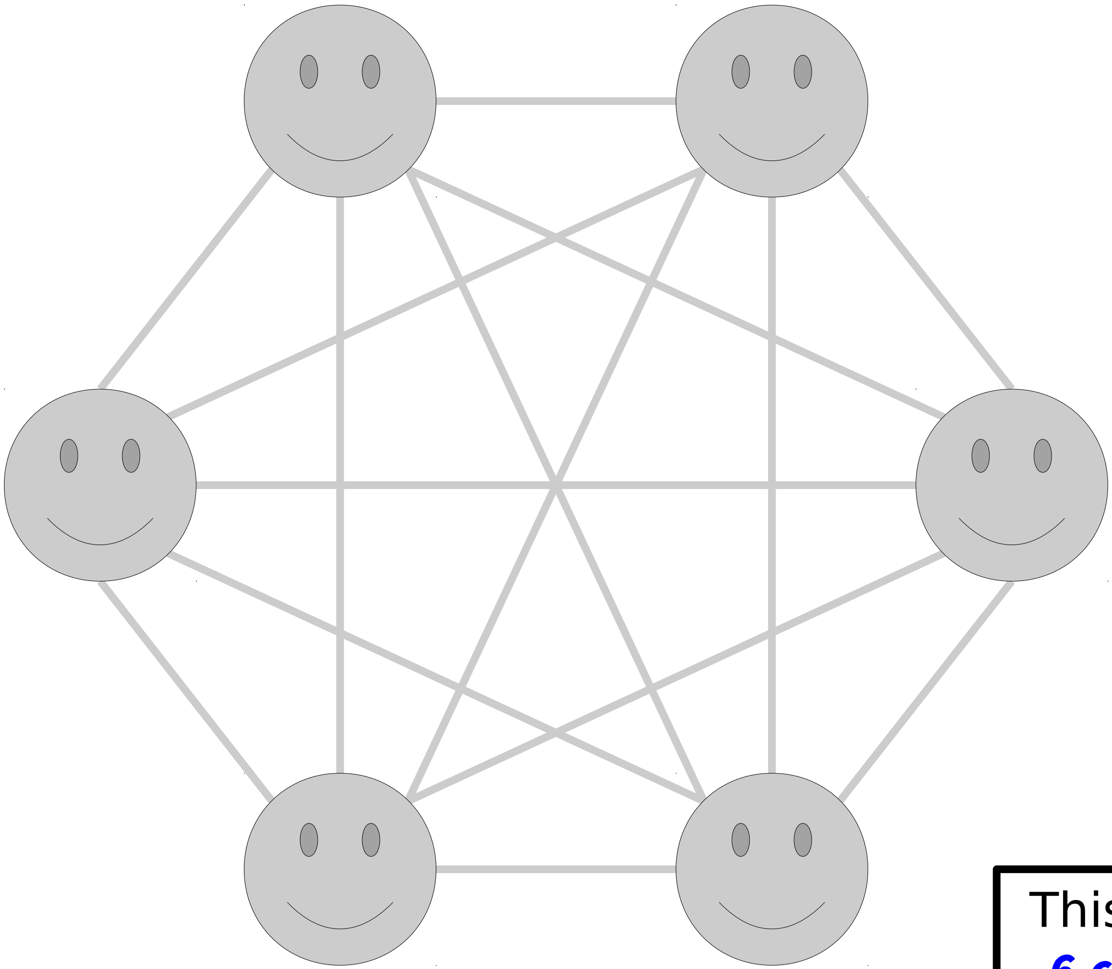




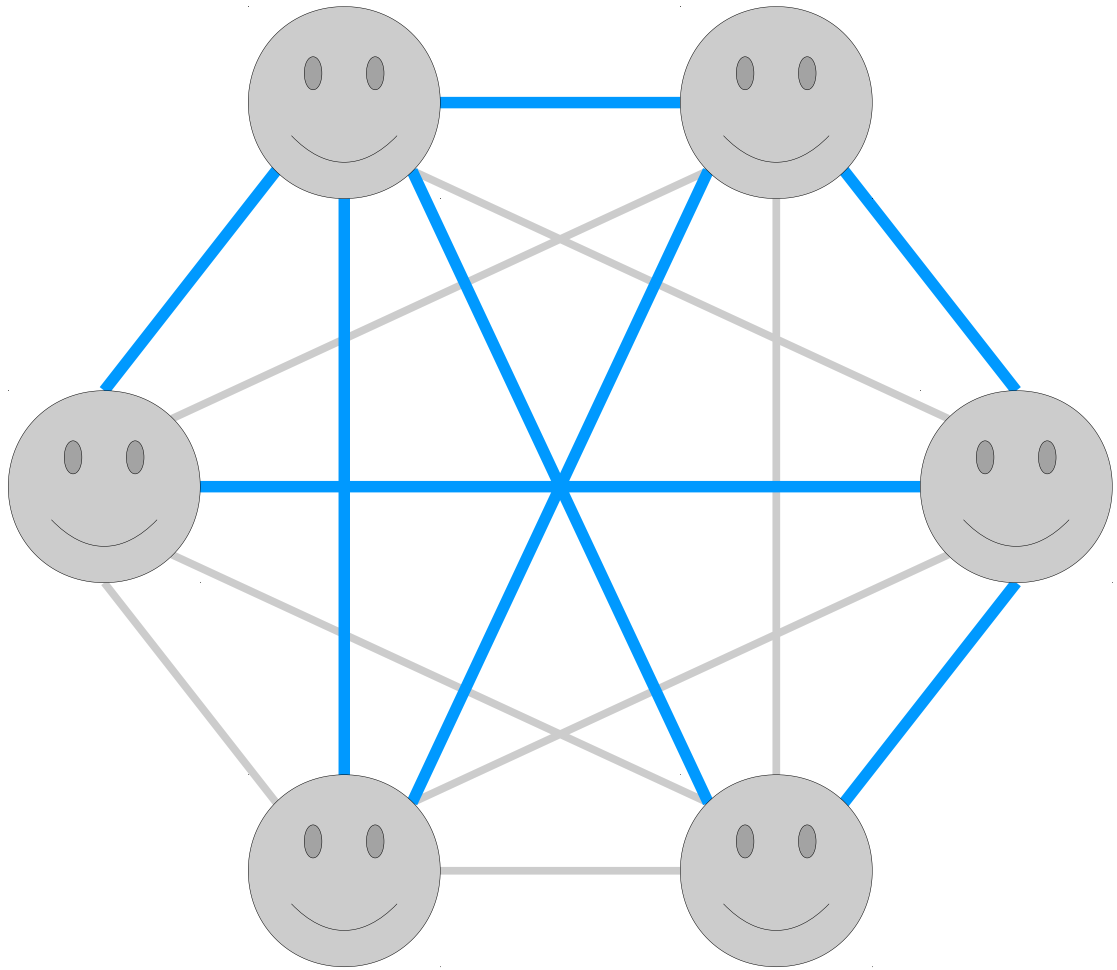


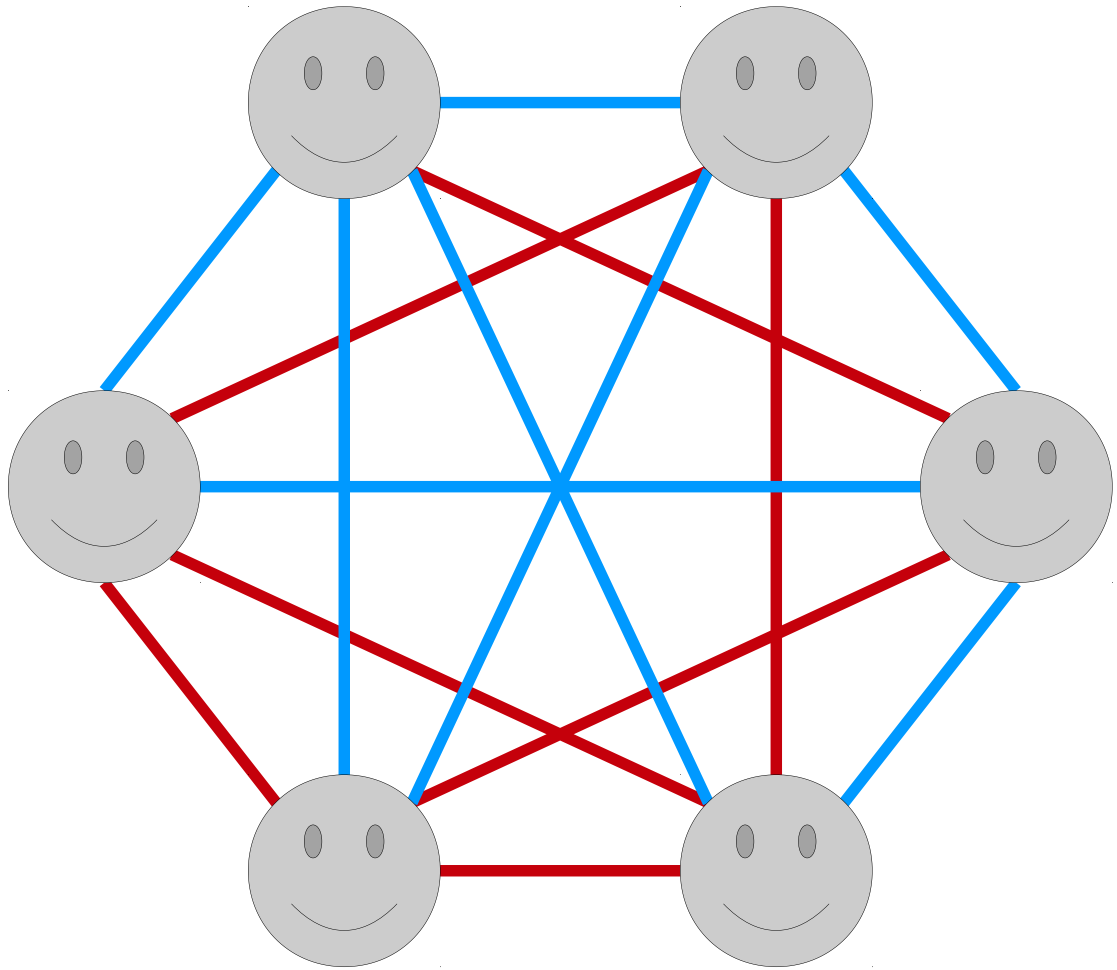


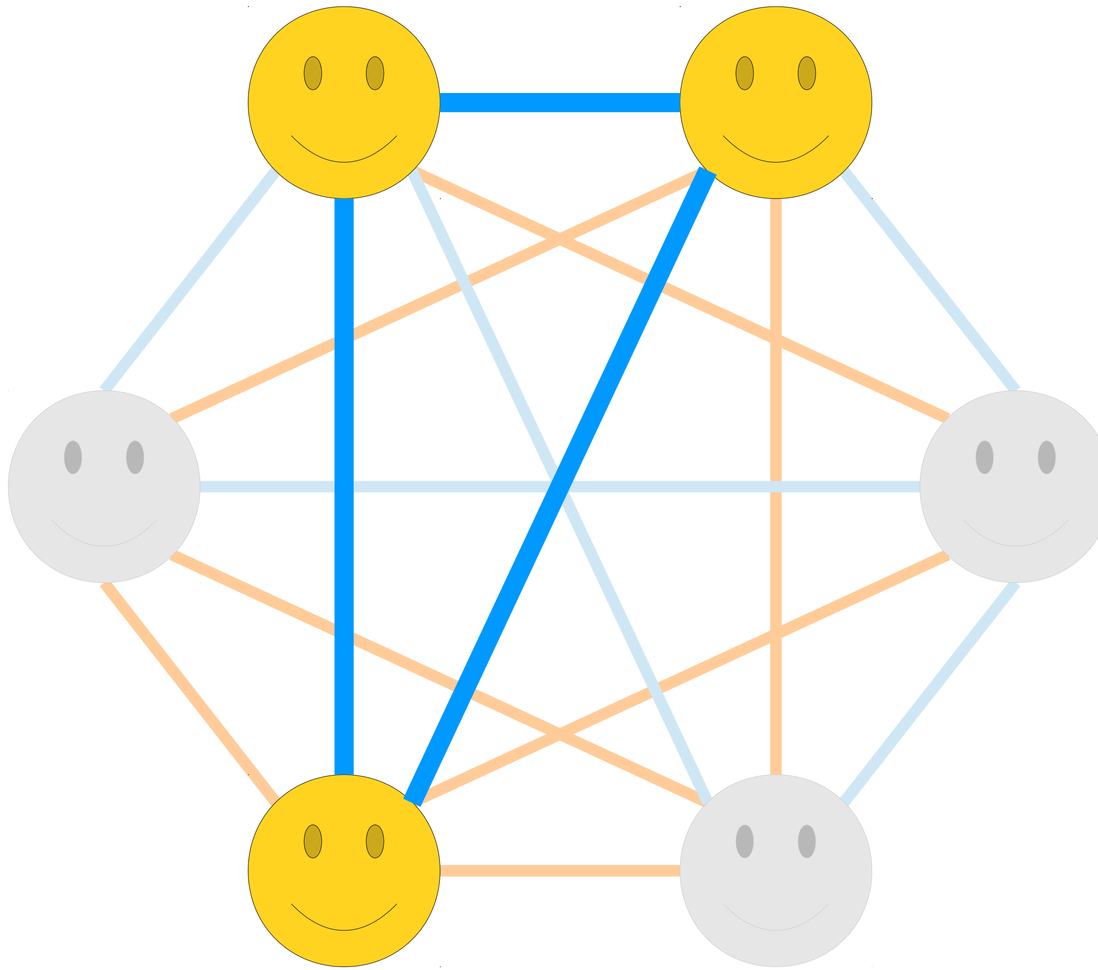


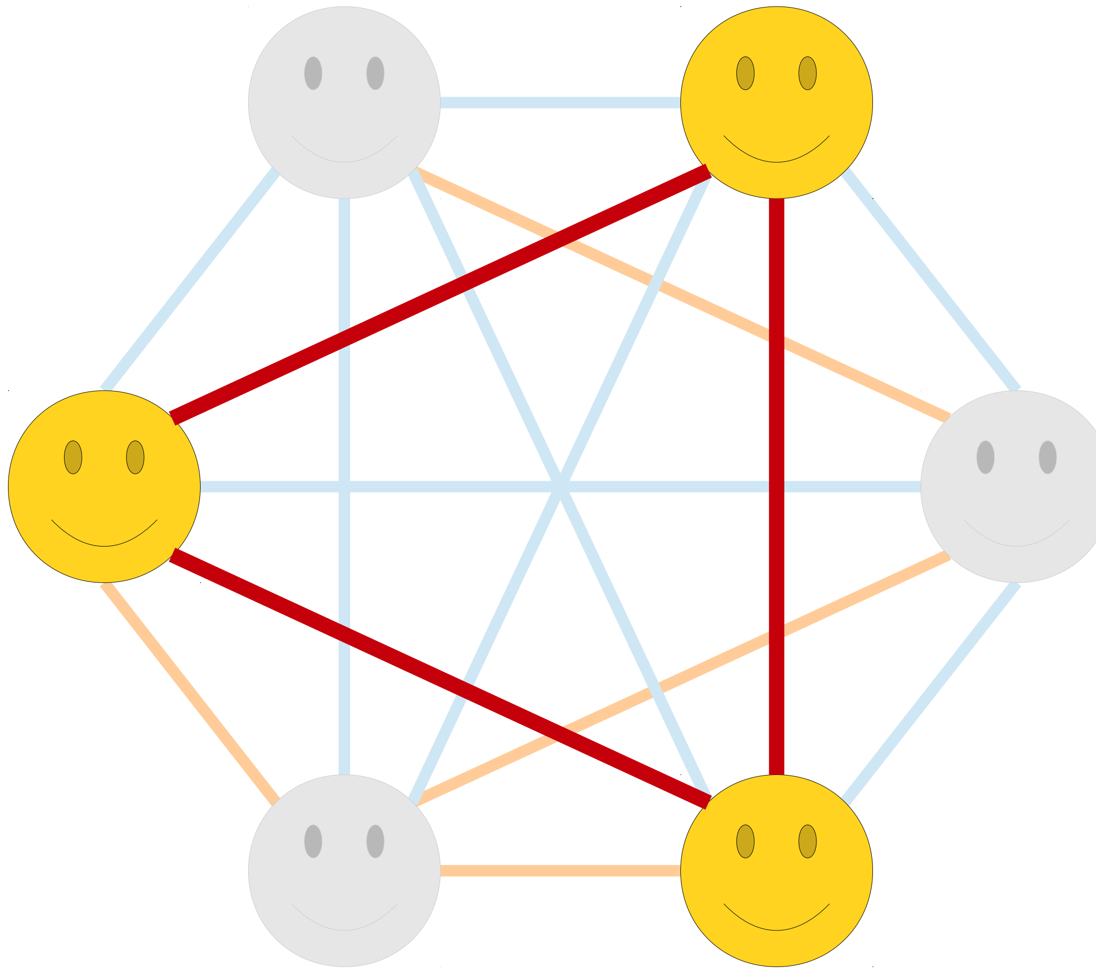


This graph is called a ***6-clique***, by the way.







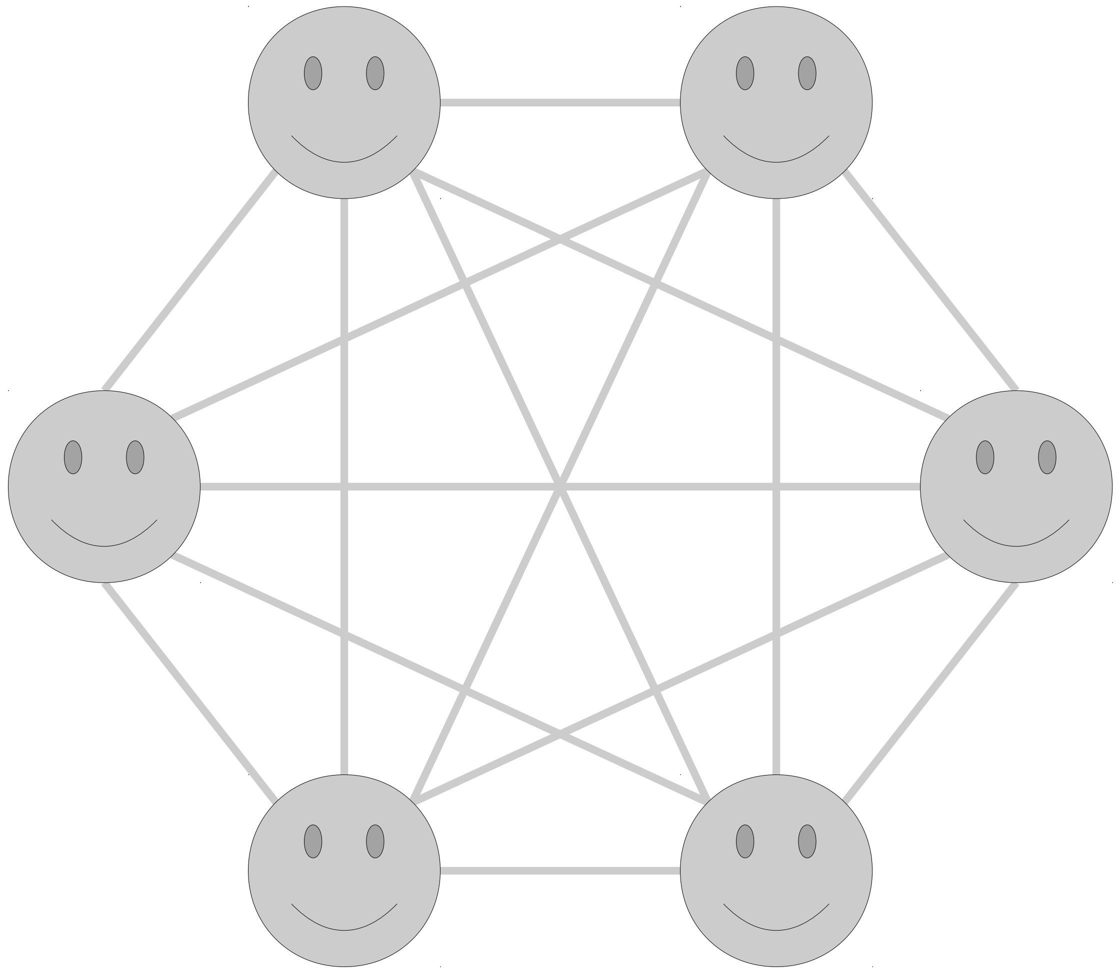


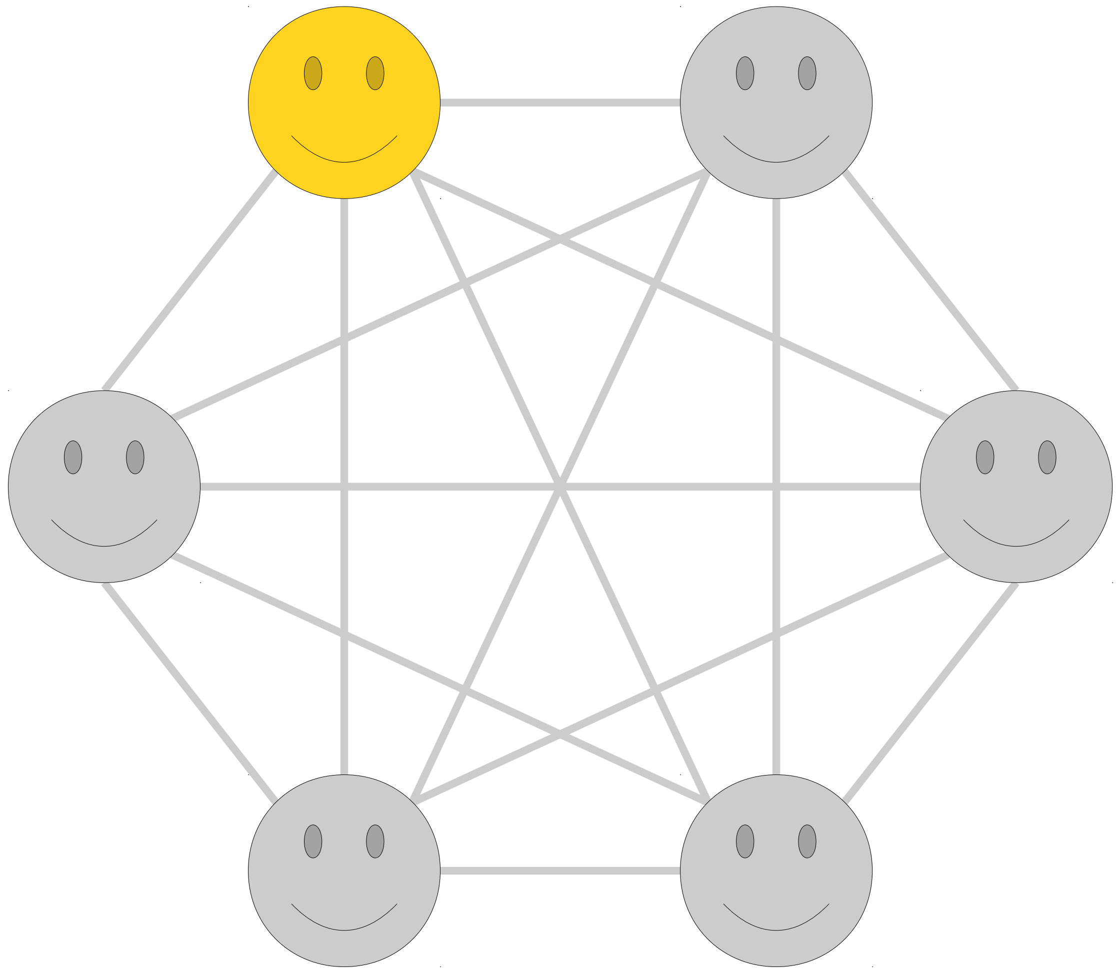
Friends and Strangers Restated

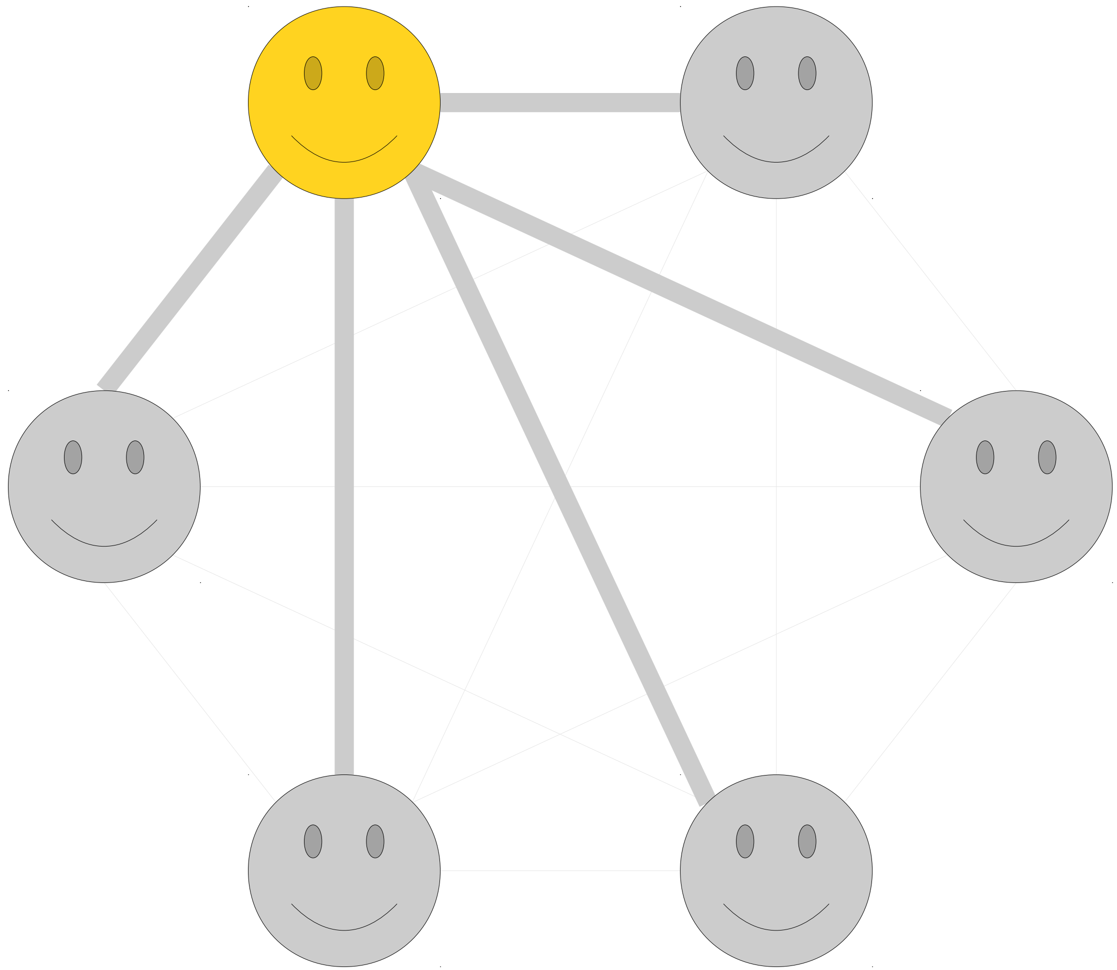
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

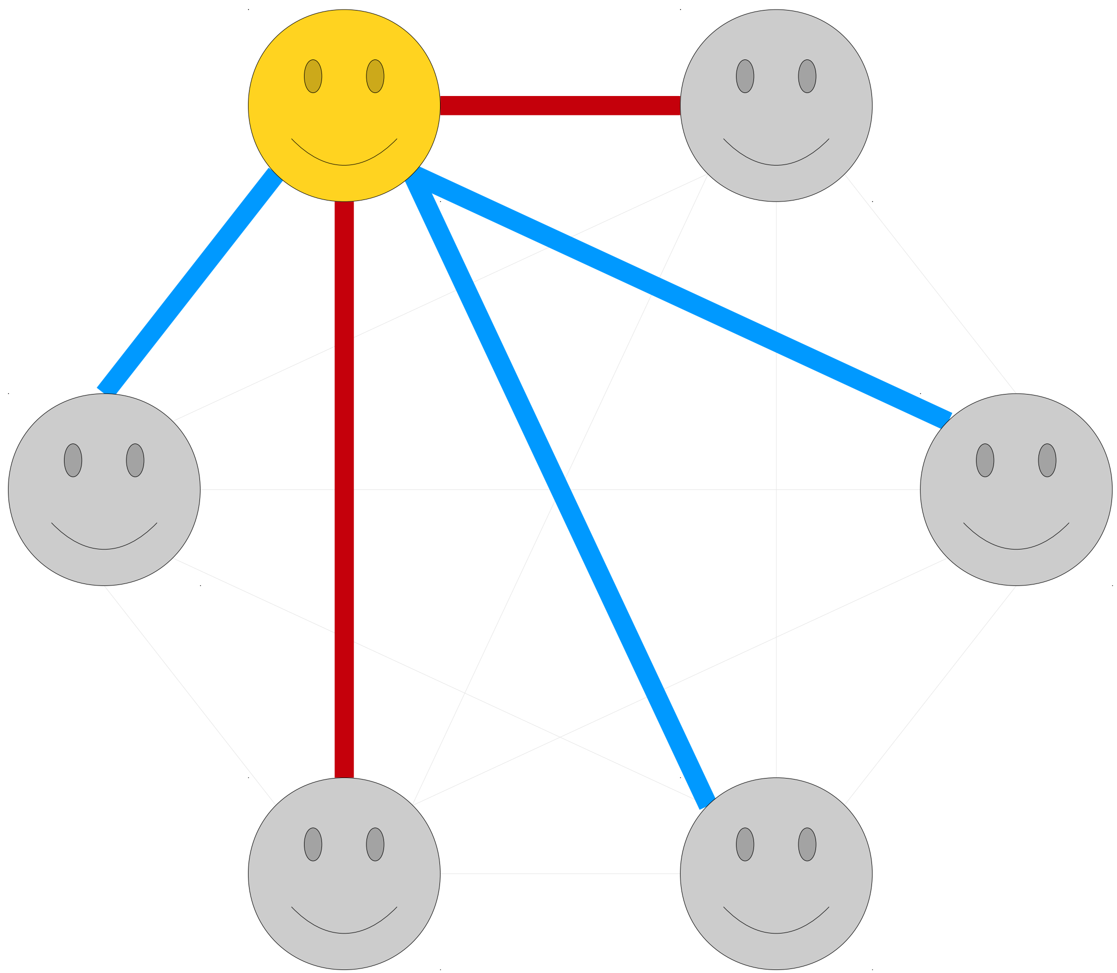
  Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

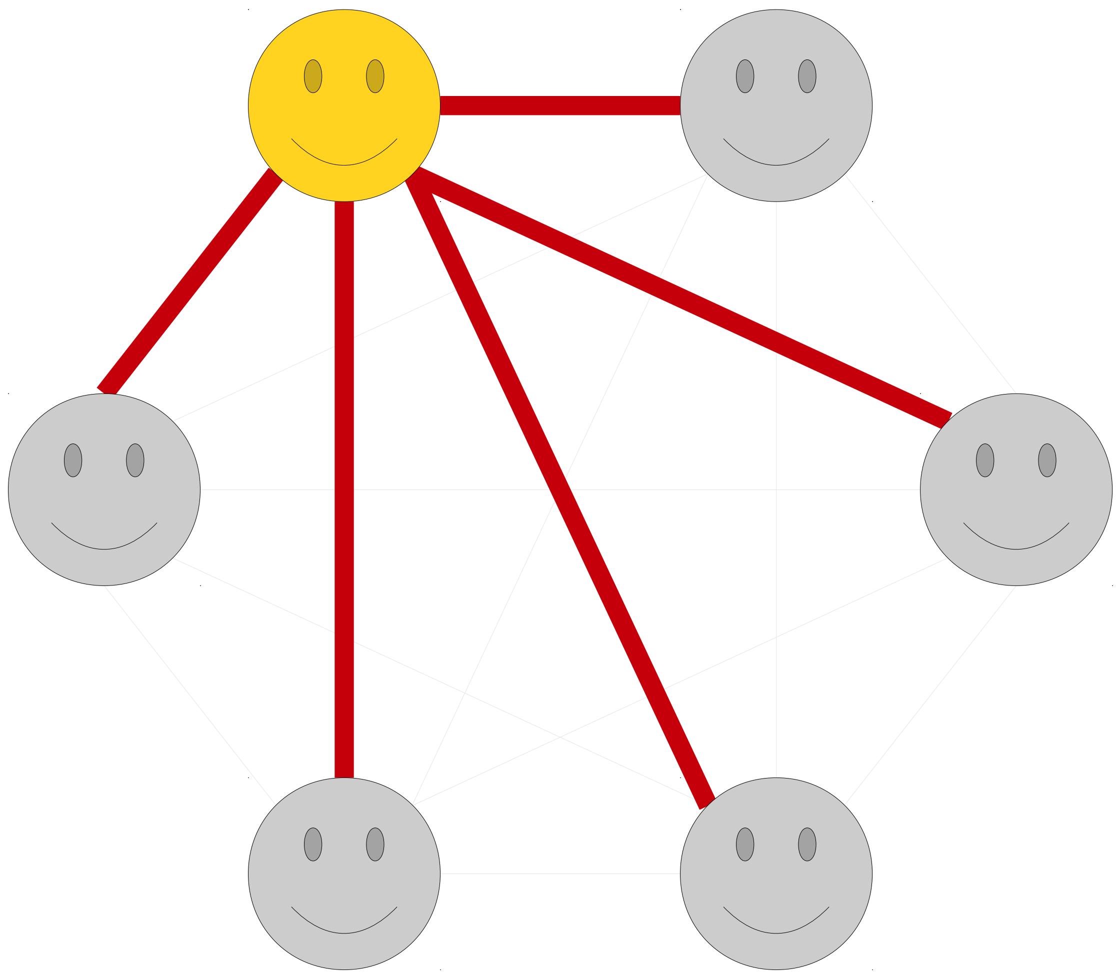
- How can we prove this?

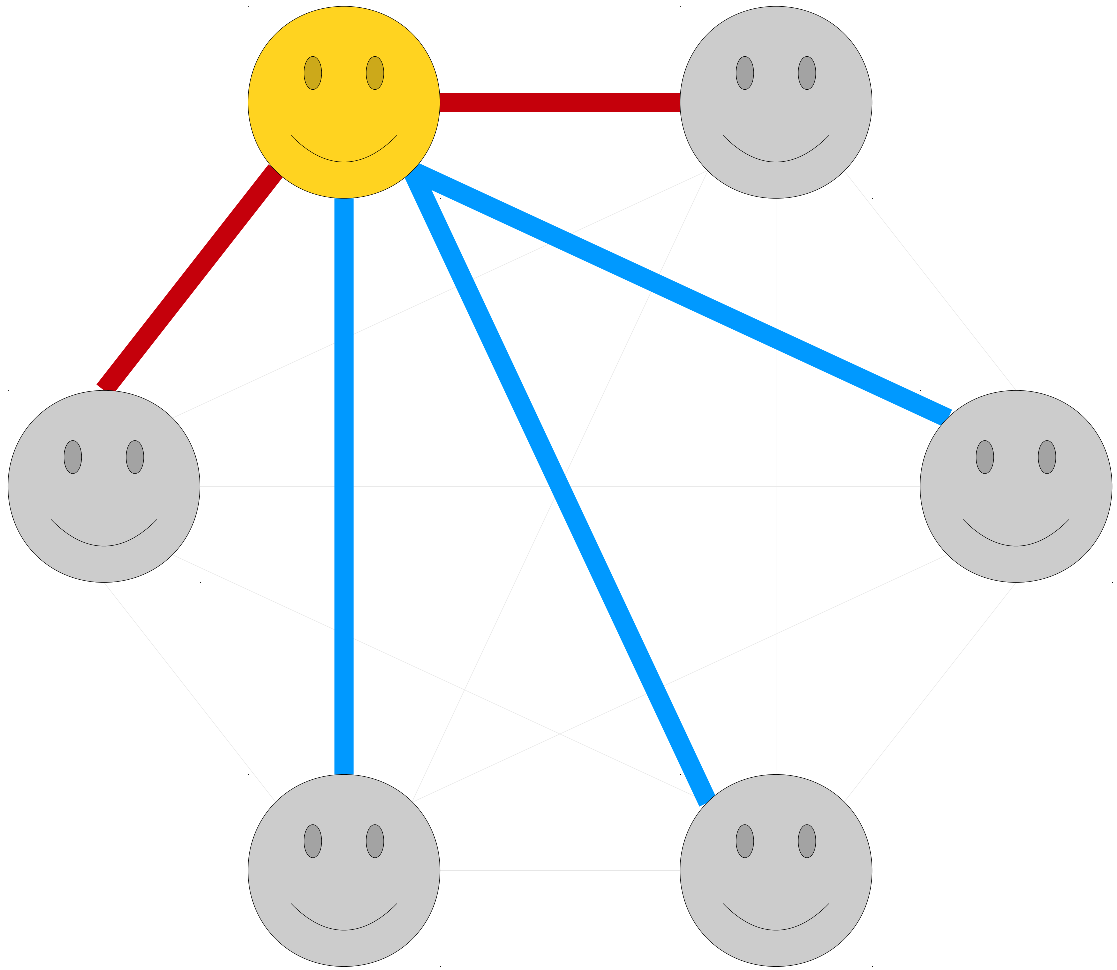


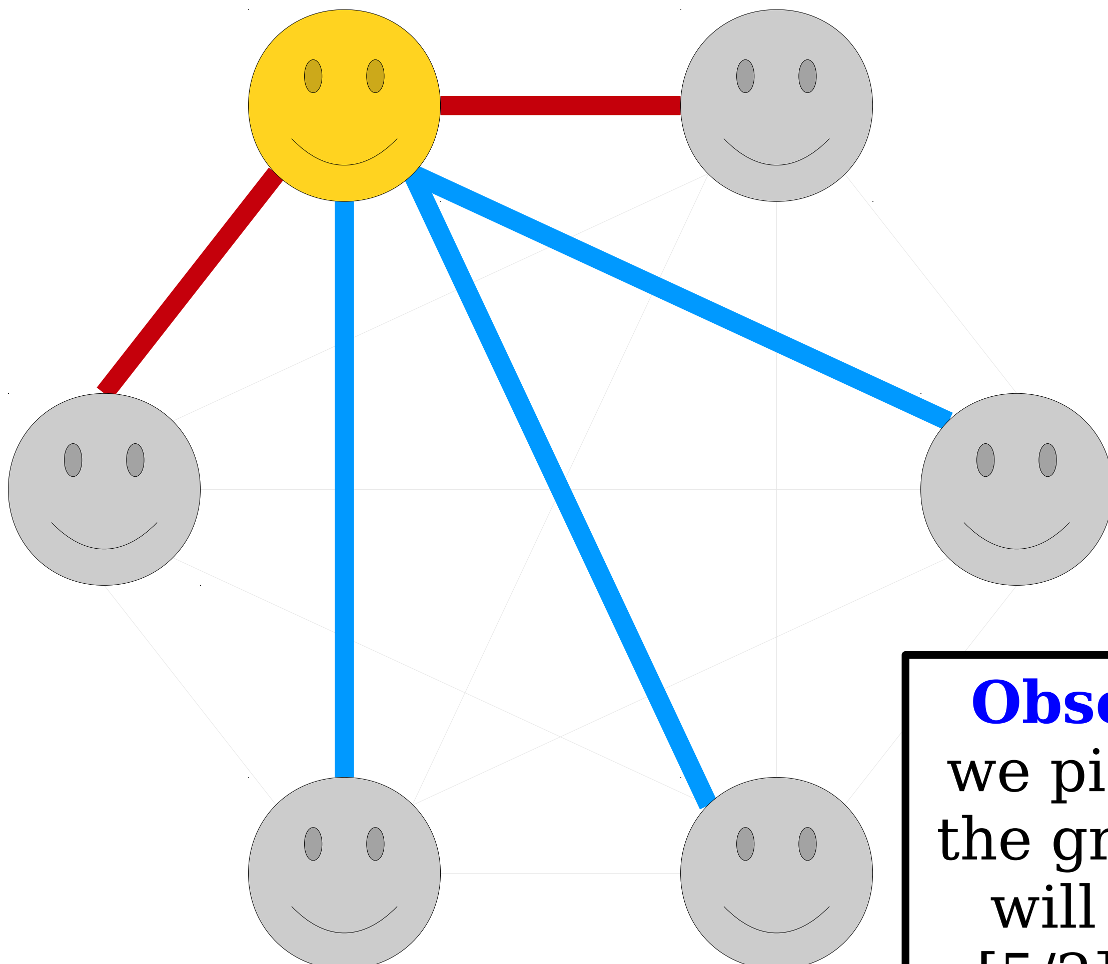




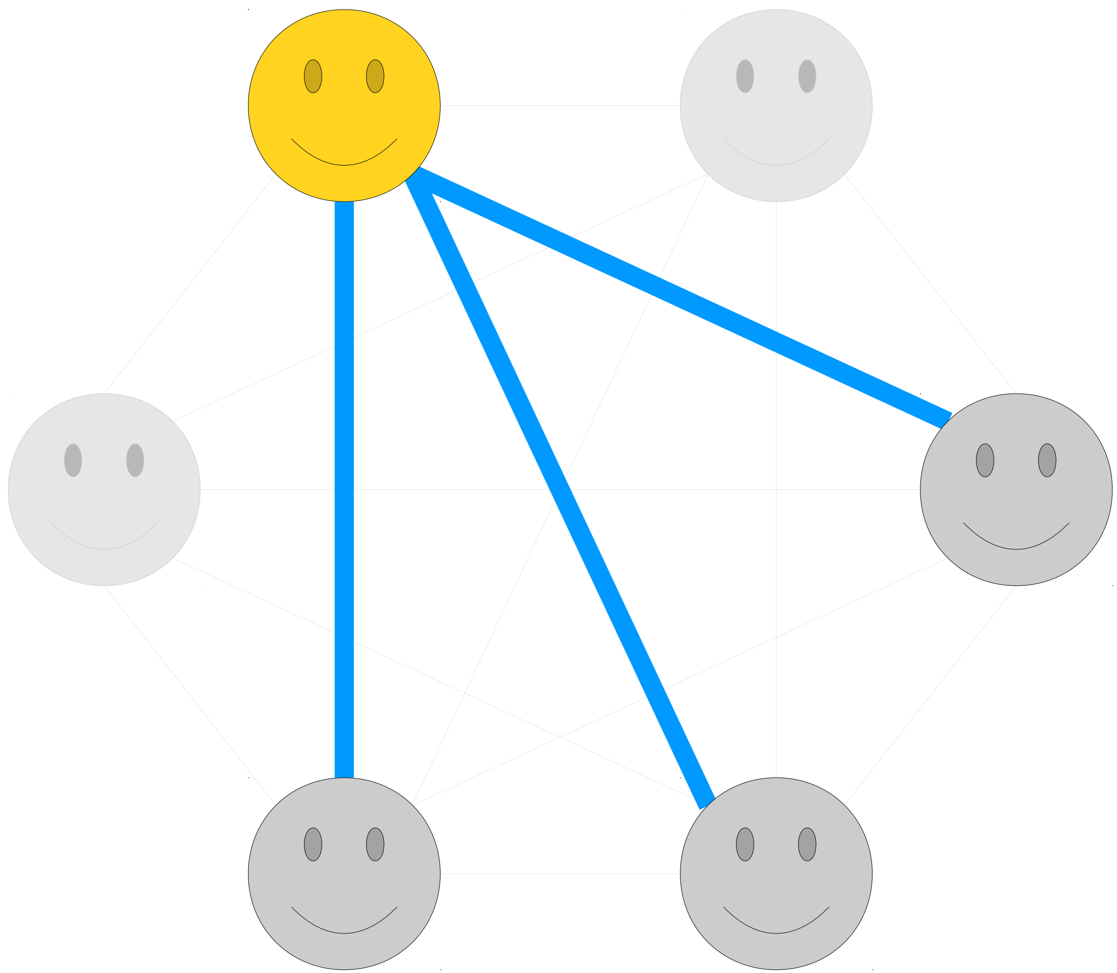


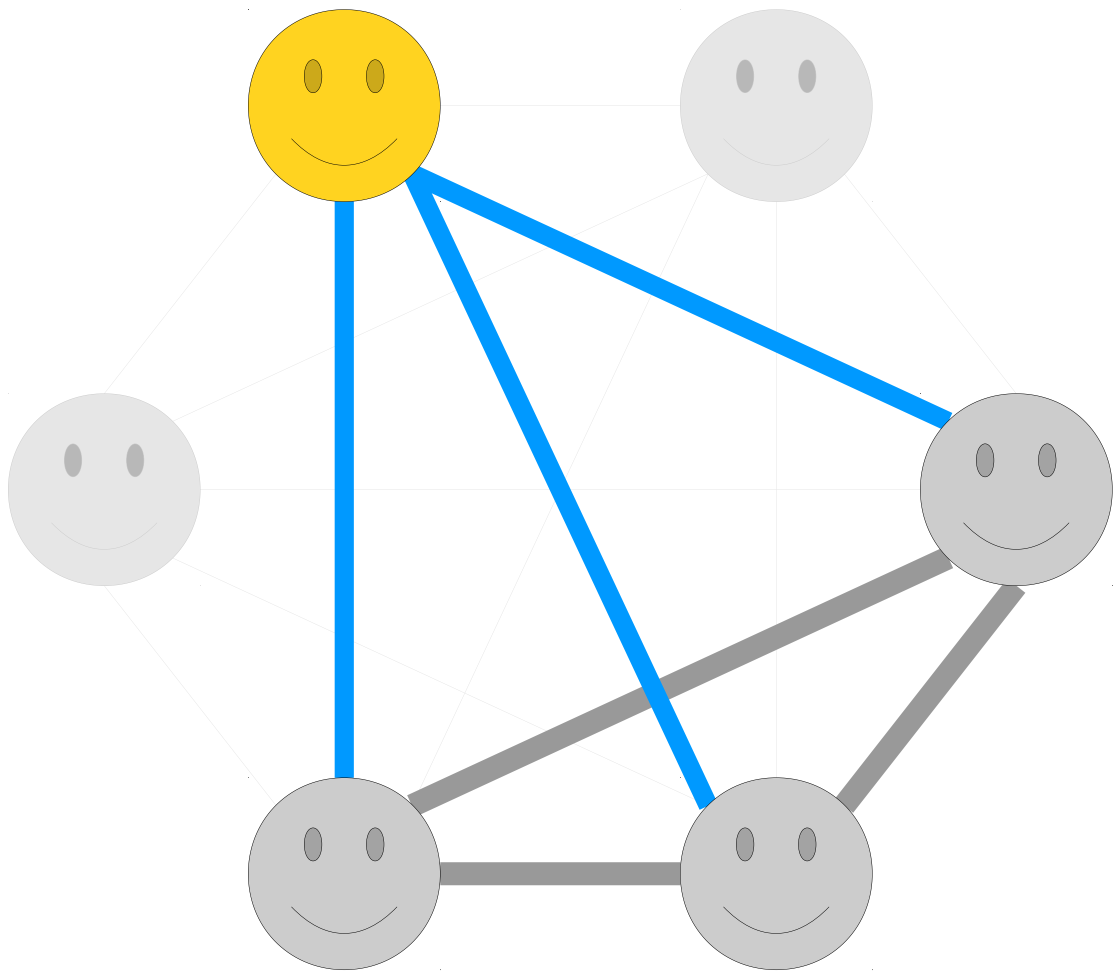


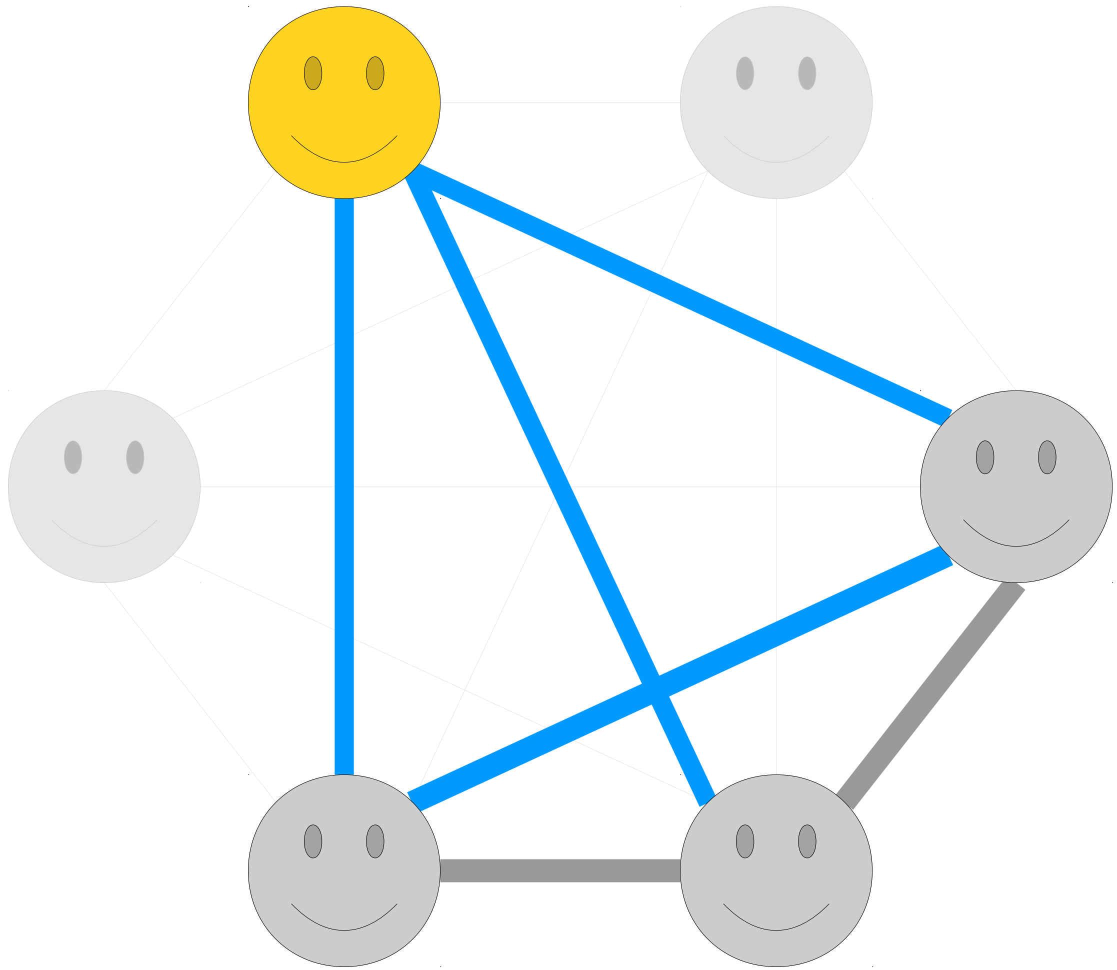


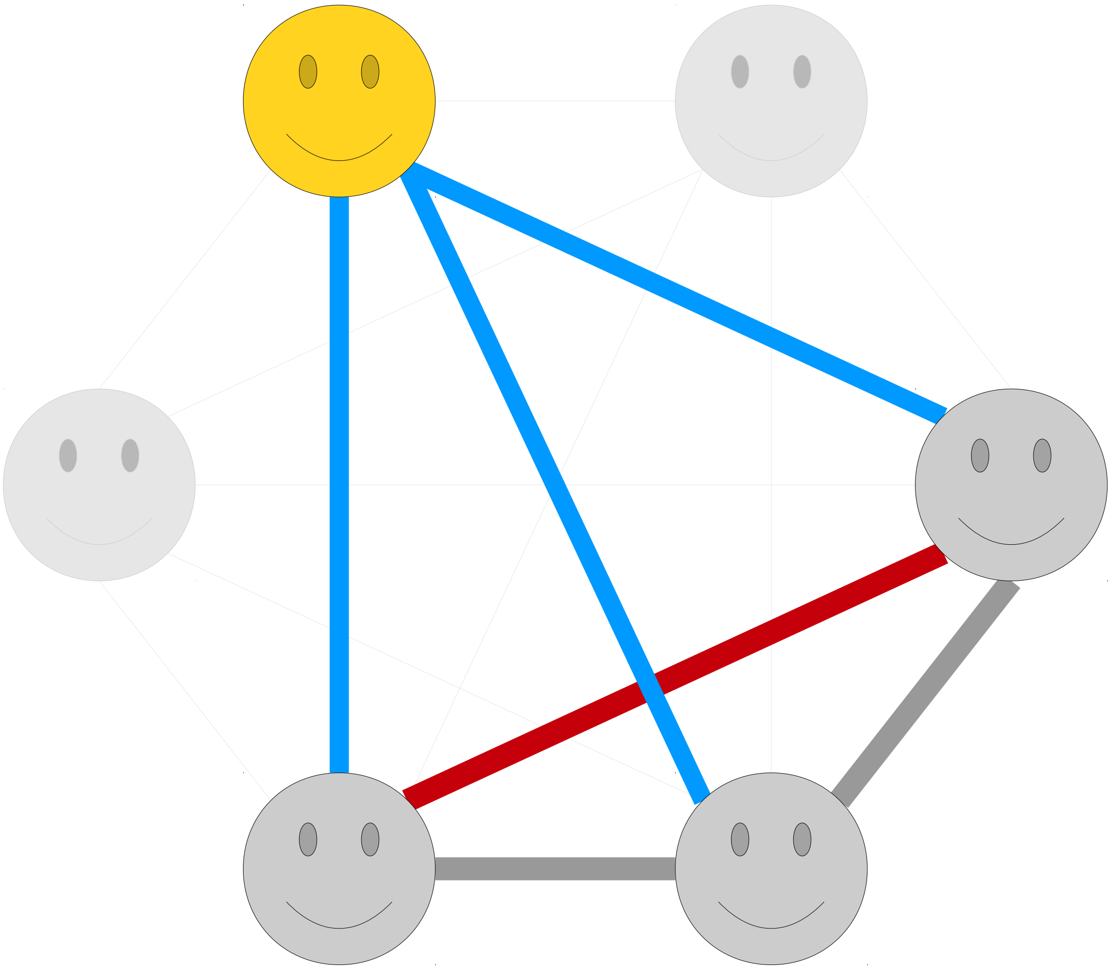


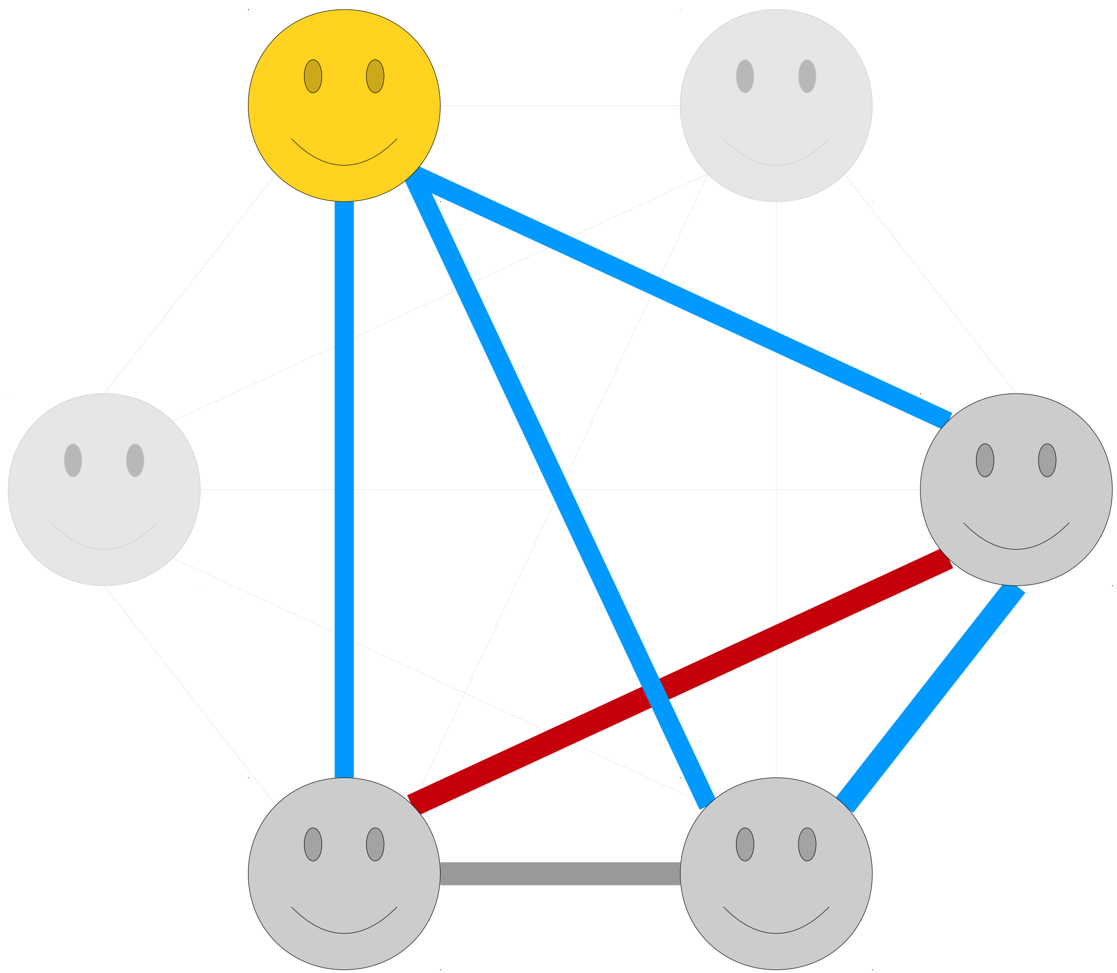
Observation 1: If we pick any node in the graph, that node will have at least ⌈5/2⌉ = 3 edges of the same color incident to it.

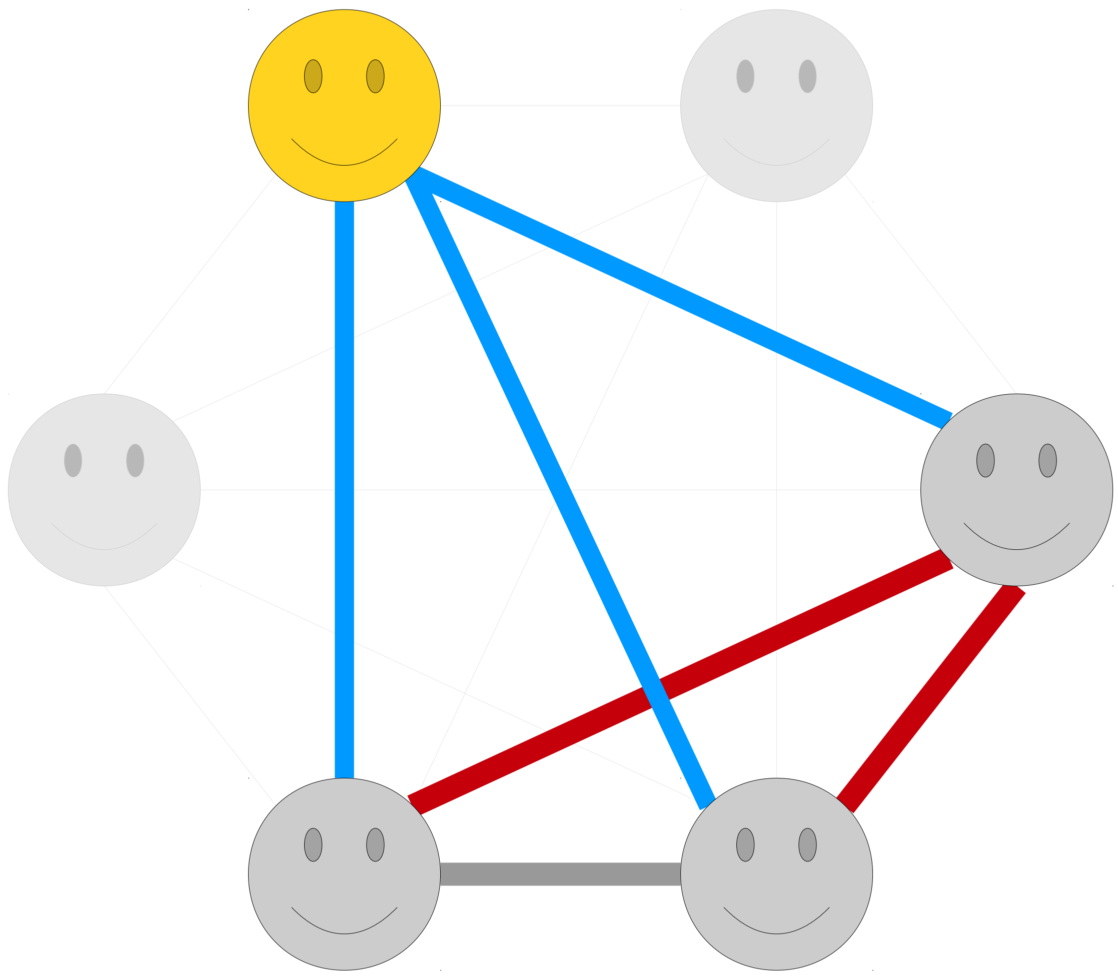


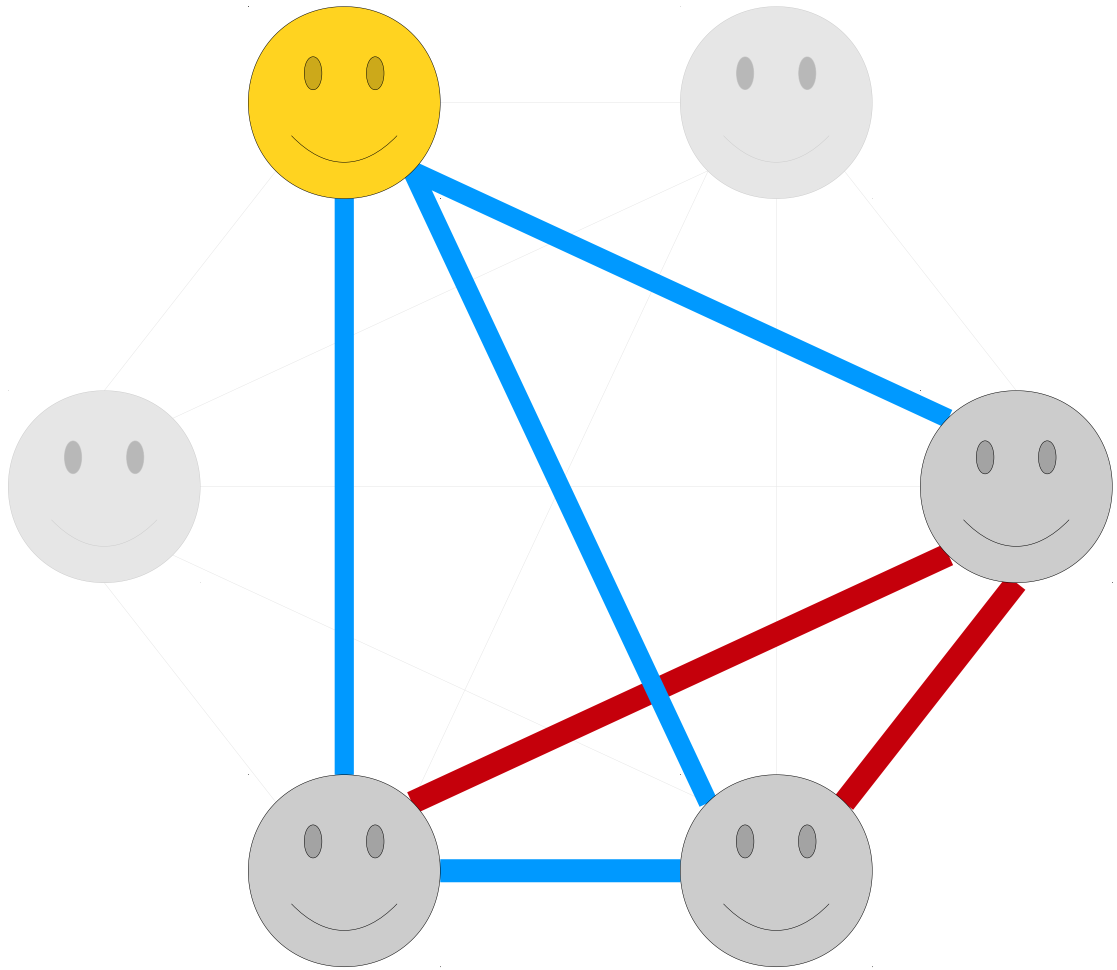


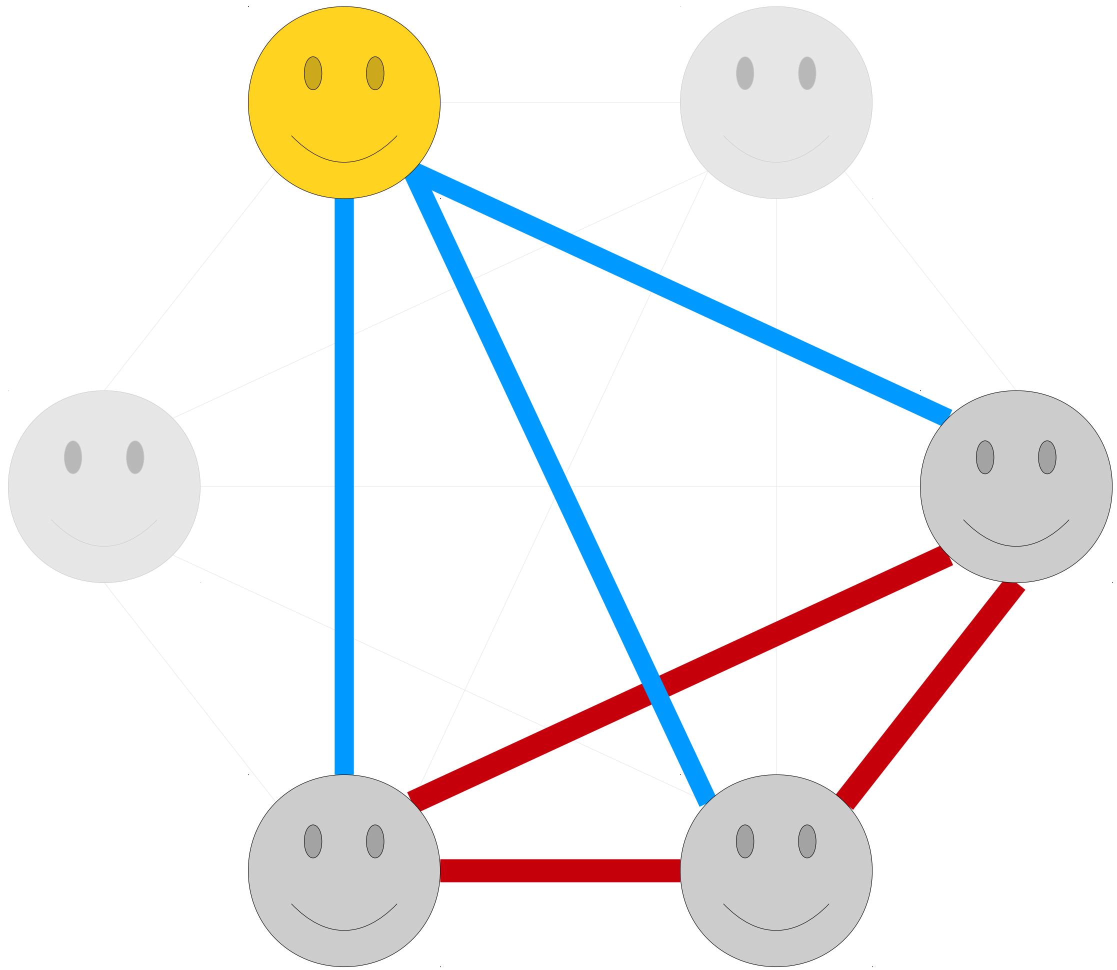












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Color the edges of the 6-clique either red or blue arbitrarily. Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2.

Let r, s, and t be three of the nodes adjacent to node x along an edge of color c_1. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1, then one of those edges plus the two edges connecting back to node x form a triangle of color c_1. Otherwise, all three of those edges are of color c_2, and they form a triangle of color c_2. Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- The proof we did is a special case of a broader result.
- ***Theorem (Ramsey's Theorem):*** For any natural number n, there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$-clique are colored red or blue, the resulting graph will contain either a red n-clique or a blue n-clique.
  - Our proof was that $R(3) \leq 6$.
- A more philosophical take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.